

AN EFFECTIVE THEORY OF GW AND FJRW INVARIANTS OF QUINTICS CALABI-YAU MANIFOLDS

HUAI-LIANG CHANG, JUN LI, WEI-PING LI, AND CHIU-CHU MELISSA LIU

ABSTRACT. This is the second part of the project toward an effective algorithm to evaluate all genus Gromov-Witten invariants of quintic Calabi-Yau threefolds. In this paper, the localization formula is derived, and algorithms toward evaluating these Gromov-Witten invariants are derived.

1. INTRODUCTION

In [CLLL], we introduced the notion of Mixed-Spin-P (MSP) fields

$$(1.1) \quad \xi = (\Sigma^c, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$$

consisting of pointed twisted curves $\Sigma^c \subset \mathcal{C}$, line bundles \mathcal{L} and \mathcal{N} , and fields $\varphi \in H^0(\mathcal{L}^{\oplus 5})$, $\rho \in H^0(\mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}}^{\log})$, and $\nu \in H^0(\mathcal{L} \otimes \mathcal{N} \oplus \mathcal{N})$. Its numerical invariants are the genus g of \mathcal{C} , the monodromy γ_i of \mathcal{L} at the marking $\Sigma_i^c \subset \Sigma^c$, and the degrees $d_0 = \deg \mathcal{L} \otimes \mathcal{N}$ and $d_\infty = \deg \mathcal{N}$, subject to certain non-degeneracy constraints.

In the same paper, we constructed the moduli $\mathcal{W}_{g,\gamma,\mathbf{d}}$ of the MSP fields of data g , $\gamma = (\gamma_1, \dots, \gamma_\ell)$ and $\mathbf{d} = (d_0, d_\infty)$, and proved that they are \mathbb{G}_m -DM stacks, have \mathbb{G}_m -equivariant relative perfect obstruction theories, and admit \mathbb{G}_m -invariant cosections (of their obstruction sheaves) with proper degeneracy loci. Applying the cosection localized virtual cycle construction, we obtain (properly supported) \mathbb{G}_m -equivariant cycles $[\mathcal{W}_{g,\gamma,\mathbf{d}}]_{\text{loc}}^{\text{vir}}$.

In this paper, we will derive a class of vanishings using these cycles. Applying the virtual localization formula to these vanishings, we find polynomial relations among the GW invariants and the FJRW invariants of Fermat quintic polynomials:

$$(1.2) \quad \sum_{\Gamma} \text{res}_{\mathfrak{t}=0} \left(\mathfrak{t}^{\delta(g,\gamma,\mathbf{d})-1} \cdot \frac{[(\mathcal{W}_{g,\gamma,\mathbf{d}})_{\Gamma}]_{\text{loc}}^{\mathbb{G}_m}}{e(N_{(\mathcal{W}_{g,\gamma,\mathbf{d}})_{\Gamma}}^{\mathbb{G}_m}/\mathcal{W}_{g,\gamma,\mathbf{d}})} \right)_0 = 0, \text{ when } \delta(g,\gamma,\mathbf{d}) > 0.$$

(See §2.2 for the notations on Γ and §2.4 for the notations for $(\mathcal{W}_{g,\gamma,\mathbf{d}})_{\Gamma}$.) Here \mathfrak{t} is the generator in $A_*^{\mathbb{G}_m}(B\mathbb{G}_m) = \mathbb{Q}[\mathfrak{t}]$, and $\delta(g,\gamma,\mathbf{d})$ is the virtual dimension of $\mathcal{W}_{g,\gamma,\mathbf{d}}$.

These relations provide an effective algorithm to evaluate all genus GW invariants of quintic threefolds, and all genus FJRW invariants of Fermat quintics

with $\frac{2}{5}$ -insertions, provided that a range of “initial” GW invariants of quintic threefolds and FJRW invariants of the Fermat quintics are known.

More precisely, let $N_{g,d}$ be the genus g degree d GW invariants of the quintic threefold; let $\mathbf{w}_5 = x_1^5 + \cdots + x_5^5$, and let $\Theta_{g,k}$ be the genus g FJRW invariants of $([\mathbb{C}^5/\mathbb{Z}_5], \mathbf{w}_5)$ with k many $\frac{2}{5}$ -insertions. (A $\frac{2}{5}$ -insertion is a marking $\Sigma_i \subset \Sigma^c$ so that the monodromy representation of \mathcal{L} along Σ_i is ζ_5^2 , where $\zeta_5 = \exp(\frac{2\pi\sqrt{-1}}{5})$.) For more details, see Subsection 2.1.)

Using relations (1.2), we obtain

Theorem 1.1. *The relations (1.2) with $\gamma = \emptyset$ and $d_\infty = 0$ provide an effective algorithm to determine the GW invariants $N_{g,d}$, provided*

- (1) $N_{g,d'}$ are known for $d' < g$;
- (2) $N_{g',d'}$ are known for $g' < g$ and $d' < d$;
- (3) $\Theta_{g',k'}$ are known for $g' \leq g$ and $k' \leq 2g - 2$.

Theorem 1.2. *The relations (1.2) with $\gamma = \emptyset$, $d_0 = 0$ and $d_\infty \geq g$ provide an effective algorithm to determine the FJRW invariants $\Theta_{g,k}$, provided that $\Theta_{g',k'}$ are known for each $g' \leq g$ and $0 \leq k' < 7g' - 2$;*

In particular all genus one FJRW invariants $\Theta_{1,k}$ for quintic singularity can be evaluated. (FJRW invariants $\Theta_{0,k}$ were calculated via Grothendieck-Riemann-Roch formula in [CR].)

We conjecture that initial values requirement (1) in Theorem 1.1 can be dropped by utilizing more relations in (1.2).

Conjecture 1.3. *By choosing d_0 and d_∞ positive, the relations (1.2) provide an effective algorithm to determine all genus GW invariant $N_{g,d}$, provided that*

- (1) $N_{g',d'}$ are known for $g' < g$, $d' \leq d$, and
- (2) $\Theta_{g',k'}$ are known for $g' \leq g$ and $k' < 7g - 2$.

Note that the range of k' needed in $\Theta_{g',k'}$ in Theorem 1.2 and Conjecture 1.3 are substantially larger than that in Theorem 1.1. To fully determine all FJRW invariants, we propose

Conjecture 1.4. *The polynomial relations among FJRW invariants $\Theta_{g,k}$ derived using the relations (1.2) with $d_0 = 0$ and $\gamma \in (\boldsymbol{\mu}_5^{\text{nar}})^{\times \ell}$ can effectively evaluate all $\Theta_{g,k}$.*

We comment that the algorithms become effective if certain (leading) coefficients are non-zero. We have tested a few examples of the relations mentioned in Conjecture 1.3 and 1.4. They are non-zero.

This work was inspired by the work of Fan-Jaris-Ruan on their analytic construction of FJRW invariants [FJR1], and by Witten’s vision that “Calabi-Yau and Landau-Ginzburg (are) separated by a true phase transition” [Wi]. In realizing Witten’s vision, we introduced a new field ν , which is our attempt to

“quantize” the Witten’s parameter in his phase transition between Calabi-Yau and Landau-Ginzberg theories.

After introducing the GW invariants of stable maps with p -fields and proving their equivalence with the GW invariants of quintic threefolds by the first two named authors [CL1], our goal is to search for a geometric theory of wall crossings envisioned by Witten. In [CLL], the first three named authors (re)reconstructed the (narrow) FJRW invariants using cosection localized virtual cycles, and in [CLLL] the authors introduced the theory of Mixed-Spin-P fields. This paper sets up the foundation for applying MSP field theory to study GW and FJRW invariants of Fermat quintics.

There are fresh attempts to understand all genus GW invariants of quintics; one of them by Fan-Jarvis-Ruan [FJR3] in which they proposed a version of GLSM theory. (See also the work of Tian-Xu [TX1, TX2].) We comment that though the MSP fields theory resembles the case [FJR3, Example 4.2.23], indeed [FJR3] and the MSP field theory are different approaches to CY/LG correspondence, which will be clarified in Subsection 5.8. The work of Choi and Kiem [ChK] introduced additional δ -stability to study wall-crossings, toward all genus GW invariants of quintic CY threefolds.

This paper is organized as follows. In §2, after reviewing the basic properties of MSP fields and their moduli spaces, we work out the the topological characterization of a partition of the \mathbb{C}^* -fixed locus of the moduli space. In §3, we work out the cosection localized virtual cycle of the fixed locus. The data in the virtual localization formula associated to the moving parts will be worked out in the following section. In §5, we derive a vanishing, and using this relation to prove the main theorems of this paper.

2. MODULI OF T -INVARIANT MSP-FIELDS

In this section, we apply the cosection localized version of the virtual localization formula of [GP] to the cycle $[\mathcal{W}_{g,\gamma,\mathbf{d}}]_{\text{loc}}^{\text{vir}}$ to obtain the vanishing (1.2). The proof of this version of virtual localization formula is in [CKL]. In this paper, all twisted curves are nodal curves.

2.1. Definition of MSP fields. Let $\mu_5 \leq \mathbb{G}_m$ be the subgroup of 5th-roots of unity, generated by $\zeta_5 = \exp(\frac{2\pi\sqrt{-1}}{5})$. Let

$$(2.1) \quad \mu_5^{\text{bro}} = \{(1, \rho), (1, \varphi)\} \cup \mu_5 \quad \text{and} \quad \mu_5^{\text{nar}} = \mu_5^{\text{bro}} - \{1\}.$$

Here $(1, \rho)$ and $(1, \varphi)$ are merely symbols, with the convention $\langle (1, \rho) \rangle = \langle (1, \varphi) \rangle = \{1\} \leq \mathbb{G}_m$. We call $\gamma \in (\mu_5^{\text{bro}})^{\times \ell}$ broad, and call $\gamma \in (\mu_5^{\text{nar}})^{\times \ell}$ narrow. We call an ℓ -pointed twisted curve $\Sigma^{\mathbb{C}} \subset \mathcal{C}$ γ -pointed if $\Sigma_i^{\mathbb{C}}$ is labeled by γ_i . Let

$$\omega_{\mathcal{C}/S}^{\log} = \omega_{\mathcal{C}/S}(\Sigma^{\mathbb{C}}), \quad \text{and} \quad \Sigma_{\alpha}^{\mathbb{C}} = \coprod_{\gamma_i = \alpha} \Sigma_i^{\mathbb{C}}$$

for $\alpha \in \mu_5^{\text{nar}}$ or μ_5^{bro} .

Definition 2.1. A (g, γ, \mathbf{d}) MSP-field is a collection as in (1.1) such that

- (1) $\cup_{i=1}^\ell \Sigma_i^{\mathcal{C}} = \Sigma^{\mathcal{C}} \subset \mathcal{C}$ is a genus g , γ -pointed twisted curve such that the i -th marking $\Sigma_i^{\mathcal{C}}$ is banded by the group $\langle \gamma_i \rangle \leq \mathbb{G}_m$;
- (2) \mathcal{L} and \mathcal{N} are invertible sheaves on \mathcal{C} , $\mathcal{L} \oplus \mathcal{N}$ representable, $\deg \mathcal{L} \otimes \mathcal{N} = d_0$, $\deg \mathcal{N} = d_\infty$, and the monodromy of \mathcal{L} along $\Sigma_i^{\mathcal{C}}$ is γ_i when $\langle \gamma_i \rangle \neq \langle 1 \rangle$;
- (3) $\nu = (\nu_1, \nu_2) \in H^0(\mathcal{L} \otimes \mathcal{N}) \oplus H^0(\mathcal{N})$, and (ν_1, ν_2) is nowhere zero;
- (4) $\varphi = (\varphi_1, \dots, \varphi_5) \in H^0(\mathcal{L})^{\oplus 5}$, (φ, ν_1) is nowhere zero, and $\varphi|_{\Sigma_{(1, \varphi)}^{\mathcal{C}}} = 0$;
- (5) $\rho \in H^0(\mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}/S}^{\log})$, (ρ, ν_2) is nowhere zero, and $\rho|_{\Sigma_{(1, \rho)}^{\mathcal{C}}} = 0$.

We call ξ broad if γ is broad (i.e. $\gamma \in (\mu_5^{\text{bro}})^{\times \ell}$); we call ξ narrow if γ is narrow (i.e. $\gamma \in (\mu_5^{\text{nar}})^{\times \ell}$).

We remark that the distinction between $(1, \varphi)$ and $(1, \rho)$ lies in item (4) and (5). When $\gamma_i = (1, \varphi)$ or $(1, \rho)$, by the representable requirement, $\Sigma_i^{\mathcal{C}}$ is a scheme point.

We add that a $\frac{2}{5}$ -insertion is a marking with monodromy representation the multiplication by ζ_5^2 . A local example is as follows. Consider $\mathcal{C} = [\mathbb{A}^1 / \mu_5]$, where μ_5 acts on $\mathbb{A}^1 = \text{Spec } \mathbb{C}[x]$ via $\zeta_5 \cdot x = \zeta_5^{-1}x$. Then the $\mathcal{O}_{\mathcal{C}}$ -module $x^{-2}\mathbb{C}[x]$ has monodromy representation via multiplication by ζ_5^2 .

Throughout this paper, unless otherwise mentioned, for any closed point $\xi \in \mathcal{W}_{g, \gamma, \mathbf{d}}$ we understand that ξ is $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \dots)$ given in (1.1).

2.2. Decorated graphs of torus fixed MSP fields. By the main theorem of [CLLL], the category $\mathcal{W}_{g, \gamma, \mathbf{d}}$ of families of MSP-fields of data (g, γ, \mathbf{d}) is a separated DM stack. Let $T = \mathbb{G}_m$, and endow $\mathcal{W}_{g, \gamma, \mathbf{d}}$ a T -structure, via

$$(2.2) \quad t \cdot (\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu_1, \nu_2) = (\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, t\nu_1, \nu_2).$$

In this subsection, we study the T -fixed locus $(\mathcal{W}_{g, \gamma, \mathbf{d}})^T \subset \mathcal{W}_{g, \gamma, \mathbf{d}}$.

In the following, we fix a (g, γ, \mathbf{d}) . For notational simplicity, whenever (g, γ, \mathbf{d}) is understood, we will adopt the convention

$$(2.3) \quad \mathcal{W} = \mathcal{W}_{g, \gamma, \mathbf{d}}, \quad \mathcal{W}^- = \mathcal{W}_{g, \gamma, \mathbf{d}}^-, \quad \text{and} \quad \mathcal{W}^T = (\mathcal{W}_{g, \gamma, \mathbf{d}})^T.$$

Let $\xi \in \mathcal{W}^T$. Following the definition of T fixed points, and after a standard algebraic argument, we conclude that there is a homomorphism h and T -linearizations τ and τ' (satisfying the obvious relations) as follows

$$h : T \rightarrow \text{Aut}(\mathcal{C}, \Sigma^{\mathcal{C}}); \quad \tau_t : h_{t*}\mathcal{L} \rightarrow \mathcal{L} \quad \text{and} \quad \tau'_t : h_{t*}\mathcal{N} \rightarrow \mathcal{N},$$

such that for a $q \in \mathbb{Z}_+$,

$$(2.4) \quad t \cdot (\varphi, \rho, \nu_1, \nu_2) = (\tau_t, \tau'_t)(h_{t*}\varphi, h_{t*}\rho, t^q h_{t*}\nu_1, h_{t*}\nu_2), \quad t \in T.$$

We call such T -actions and linearizations those induced from $\xi \in \mathcal{W}^T$.

Using that $\xi \in \mathcal{W}^T$ is stable, one sees that such (h, τ_t, τ'_t) is unique.

Definition 2.2. Given $\xi \in \mathcal{W}^T$, let $\mathcal{C}_0 = \mathcal{C} \cap (\nu_1 = 0)_{\text{red}}$, $\mathcal{C}_\infty = \mathcal{C} \cap (\nu_2 = 0)_{\text{red}}$, $\mathcal{C}_1 = \mathcal{C} \cap (\rho = \varphi = 0)_{\text{red}}$, \mathcal{C}_{01} (resp. $\mathcal{C}_{1\infty}$) be the union of irreducible components of $\mathcal{C} - \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_\infty$ in $(\rho = 0)$ (resp. in $(\varphi = 0)$), and $\mathcal{C}_{0\infty}$ be the union of irreducible components of \mathcal{C} not contained in $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_\infty \cup \mathcal{C}_{01} \cup \mathcal{C}_{1\infty}$.

As argued in [CLLL], \mathcal{C}_0 , \mathcal{C}_1 and \mathcal{C}_∞ are mutually disjoint. By definition, all \mathcal{C}_{01} , $\mathcal{C}_{0\infty}$ and $\mathcal{C}_{1\infty}$ are pure one-dimensional, and T -equivariant curves under h . We recall a simple lemma (cf. [CLLL, Lemma 3.2]).

Lemma 2.3. Let $T \rightarrow \text{Aut}(\mathbb{P}^1)$ be a non-trivial homomorphism of algebraic groups whose induced action on \mathbb{P}^1 fixes $[0, 1]$ and $[1, 0]$, and let L be a T -linearized line bundle on \mathbb{P}^1 such that T acts trivially on $L|_{[0,1]}$. Then

- (1) any invariant section $s \in H^0(L)^T$ with $s|_{[0,1]} = 0$ must be zero;
- (2) suppose T acts trivially on $L|_{[1,0]}$, then $L \cong \mathcal{O}_{\mathbb{P}^1}$.

Lemma 2.4. Let $\xi \in \mathcal{W}^T$. Then

- (1) the action $h : T \rightarrow \text{Aut}(\mathcal{C}, \Sigma^{\mathcal{C}})$ acts trivially on \mathcal{C}_0 , \mathcal{C}_1 and \mathcal{C}_∞ ;
- (2) every irreducible component $\mathcal{A} \subset \mathcal{C}_{01}$ (resp. $\mathcal{A} \subset \mathcal{C}_{1\infty}$; resp. $\mathcal{A} \subset \mathcal{C}_{0\infty}$) is a smooth rational twisted curve with two T -fixed points lying on \mathcal{C}_0 and \mathcal{C}_1 (resp. \mathcal{C}_1 and \mathcal{C}_∞ ; resp. \mathcal{C}_0 and \mathcal{C}_∞).
- (3) if $\xi \in \mathcal{W}^-$, then $\mathcal{C}_{0\infty} = \emptyset$.

Proof. We prove that h fixes \mathcal{C}_0 . Let $\mathcal{A} \subset \mathcal{C}_0$ be an irreducible component over which $h|_{\mathcal{A}}$ is non-trivial. Then \mathcal{A} must be a rational curve. Let $x \in \mathcal{A}$ be fixed by h . Since both $\varphi(x)$ and $\nu_2(x)$ are non-zero, $\tau_t|_x = \tau'_t|_x = \text{id}$ (notice $\nu_1|_{\mathcal{A}} = 0$). Thus Lemma 2.3 shows that the component \mathcal{A} makes ξ unstable, a contradiction. Thus h fixes \mathcal{A} . Likewise, h also fixes \mathcal{C}_1 and \mathcal{C}_∞ .

We next fix an irreducible component $\mathcal{A} \subset \mathcal{C}_{01}$ and investigate the action $h|_{\mathcal{A}}$. First, by definition of \mathcal{C}_{01} , $\nu_2|_{\mathcal{A}}$ is nowhere vanishing, thus $\mathcal{N}|_{\mathcal{A}} \cong \mathcal{O}_{\mathcal{A}}$. Let $x \in \mathcal{A}$. Suppose $h|_{\mathcal{A}}$ fixes x . It is easy to see that one of $\varphi(x)$ and $\nu_1(x)$ must be zero. If x is a general point of \mathcal{A} , then by definition of \mathcal{C}_{01} , both $\varphi(x)$ and $\nu_1(x)$ are non-zero. This proves that $h|_{\mathcal{A}}$ acts non-trivially on \mathcal{A} , implying that \mathcal{A} is rational.

We claim that $\mathcal{A} \subset \mathcal{C}_{01}$ must be smooth. Otherwise, let $x \in \mathcal{A}$ be a node of \mathcal{A} , which must be fixed by $h|_{\mathcal{A}}$. Then apply Lemma 2.3, and using that either $\nu_1(x) \neq 0$ or $\varphi(x) \neq 0$, we conclude that $\mathcal{L}|_{\mathcal{A}} \cong \mathcal{O}_{\mathcal{A}}$, which forces ξ unstable, a contradiction.

As \mathcal{A} is smooth, $h|_{\mathcal{A}}$ has two fixed points, say $x_1, x_2 \in \mathcal{A}$. Like before, if both $\varphi(x_1)$ and $\varphi(x_2)$ are non-zero, then $\mathcal{L}|_{\mathcal{A}} \cong \mathcal{O}_{\mathcal{A}}$, forcing ξ unstable, a contradiction; if both $\varphi(x_1)$ and $\varphi(x_2)$ are zero, then both $\nu_1(x_1)$ and $\nu_1(x_2)$ are non-zero, forcing $\mathcal{L}|_{\mathcal{A}} \cong \mathcal{O}_{\mathcal{A}}$ and $\varphi|_{\mathcal{A}} = 0$, implying that $\mathcal{A} \subset \mathcal{C}_1$, impossible. This proves that one of x_1 and x_2 lies in \mathcal{C}_0 and the other lies in \mathcal{C}_1 .

A similar argument shows that the parallel conclusion holds for $\mathcal{A} \subset \mathcal{C}_{1\infty}$ and $\mathcal{C}_{0\infty}$. Finally, (3) follows from that both ρ and φ are non-zero on every irreducible component of $\mathcal{C}_{0\infty}$. \square

We now associate a decorated graph to each $\xi \in \mathcal{W}^T$. For a graph Γ , besides its vertices $V(\Gamma)$, edges $E(\Gamma)$ and legs $L(\Gamma)$, the set of its flags is

$$F(\Gamma) = \{(e, v) \in E(\Gamma) \times V(\Gamma) : v \in e\}.$$

Given $\xi \in \mathcal{W}^T$, let $\pi : \mathcal{C}^{\text{nor}} \rightarrow \mathcal{C}$ be its normalization. For any $y \in \pi^{-1}(\mathcal{C}_{\text{sing}})$, we denote by γ_y the monodromy of $\pi^* \mathcal{L}$ along y .

Definition 2.5. *To each $\xi \in \mathcal{W}^T$ we associate a graph Γ_ξ as follows:*

- (1) (vertex) let $V_0(\Gamma_\xi)$, $V_1(\Gamma_\xi)$, and $V_\infty(\Gamma_\xi)$ be the set of connected components of \mathcal{C}_0 , \mathcal{C}_1 , \mathcal{C}_∞ respectively, and let $V(\Gamma_\xi)$ be their union;
- (2) (edge) let $E_0(\Gamma_\xi)$, $E_\infty(\Gamma_\xi)$ and $E_{0\infty}(\Gamma_\xi)$ be the set of irreducible components of \mathcal{C}_{01} , $\mathcal{C}_{1\infty}$ and $\mathcal{C}_{0\infty}$ respectively, and let $E(\Gamma_\xi)$ be their union;
- (3) (leg) let $L(\Gamma_\xi) \cong \{1, \dots, \ell\}$ be the ordered set of markings of $\Sigma^\mathcal{C}$, $\Sigma_i^\mathcal{C} \in L(\Gamma_\xi)$ is attached to $v \in V(\Gamma_\xi)$ if $\Sigma_i^\mathcal{C} \in \mathcal{C}_v$;
- (4) (flag) $(e, v) \in F(\Gamma_\xi)$ if and only if $\mathcal{C}_e \cap \mathcal{C}_v \neq \emptyset$.

(Here \mathcal{C}_a is the curve associated to the symbol $a \in V(\Gamma_\xi) \cup E(\Gamma_\xi)$.) We call $v \in V(\Gamma_\xi)$ stable if $\mathcal{C}_v \subset \mathcal{C}$ is 1-dimensional, otherwise unstable.

In the following, let $V^S(\Gamma_\xi) \subset V(\Gamma_\xi)$ be the set of stable vertices and $V^U(\Gamma_\xi) \subset V(\Gamma_\xi)$ be the set of unstable vertices. Given $v \in V(\Gamma_\xi)$, let $E_v = \{e \in E(\Gamma_\xi) : (e, v) \in F(\Gamma_\xi)\}$. For $v \in V^S(\Gamma_\xi)$, we define

$$\Sigma_{\text{inn}}^{\mathcal{C}_v} = \Sigma^\mathcal{C} \cap \mathcal{C}_v, \quad \Sigma_{\text{out}}^{\mathcal{C}_v} = \overline{(\mathcal{C} - \mathcal{C}_v)} \cap \mathcal{C}_v, \quad \text{and} \quad \Sigma^{\mathcal{C}_v} = \Sigma_{\text{inn}}^{\mathcal{C}_v} \cup \Sigma_{\text{out}}^{\mathcal{C}_v},$$

called the inner, the outer, and the total markings of \mathcal{C}_v , respectively.

Definition 2.6. *We endow the graph Γ_ξ a decoration as follows:*

- (a) (genus) Define $\vec{g} : V(\Gamma_\xi) \rightarrow \mathbb{Z}_{\geq 0}$ via $\vec{g}(v) = h^1(\mathcal{O}_{\mathcal{C}_v})$.
- (b) (degree) Define $\vec{d} : E(\Gamma_\xi) \cup V(\Gamma_\xi) \rightarrow \mathbb{Q}^{\oplus 2}$ via $\vec{d}(a) = (d_{0a}, d_{\infty a})$, where $d_{0a} = \deg \mathcal{L} \otimes \mathcal{N}|_{\mathcal{C}_a}$ and $d_{\infty a} = \deg \mathcal{N}|_{\mathcal{C}_a}$.
- (c) (marking) Define $\vec{S} : V(\Gamma_\xi) \rightarrow 2^{L(\Gamma_\xi)}$ via $v \mapsto S_v \subset L(\Gamma_\xi)$, where S_v is the set of $\Sigma_i^\mathcal{C} \in \mathcal{C}_v$.
- (d) (monodromy) Define $\vec{\gamma} : L(\Gamma_\xi) \rightarrow \mu_5$ via $\vec{\gamma}(\Sigma_i^\mathcal{C}) = \gamma_i$, which is the monodromy of \mathcal{L} along $\Sigma_i^\mathcal{C}$.

For $\xi \in \mathcal{W}^T$, the graph Γ_ξ is the one defined in Definition 2.5 endowed with the decoration given in this definition.

For convenience, we denote $d_a = d_{0a} - d_{\infty a}$, i.e., $d_a = \deg \mathcal{L}|_{\mathcal{C}_a}$. For $c \in \{0, 1, \infty\}$, let $V_c^S(\Gamma_\xi) = V_c(\Gamma_\xi) \cap V^S(\Gamma_\xi)$. For $v \in V(\Gamma_\xi)$, the valency of v is the cardinality of E_v : $\text{val}(v) = |E_v|$. Let $n_v = |E_v \cup S_v|$. When v is stable, as S_v and E_v parameterize the inner and outer markings of \mathcal{C}_v respectively, n_v is the number of total markings of \mathcal{C}_v .

For $v \in V_\infty^S(\Gamma_\xi)$, we decorate elements in $E_v \cup S_v$ as follows. For $a \in S_v$, we decorate it by the same γ_a that decorates $a \in L(\Gamma)$; for $e \in E_v$, we decorate it

by $\gamma_{(e,v)} := e^{-2\pi\sqrt{-1}d_e} \in \mu_5$, ($d_e = \deg \mathcal{L}|_{\mathcal{C}_e}$) which is the monodromy of $\mathcal{L}|_{\mathcal{C}_e}$ along $y_{(e,v)} = \mathcal{C}_e \cap \mathcal{C}_v$. Define

$$(2.5) \quad \gamma_v = \{\gamma_a : a \in S_v\} \cup \{\gamma_{(e,v)} : e \in E_v\}.$$

Elements in γ_v are indexed by $S_v \cup E_v$.

We form

$$(2.6) \quad V^{a,b}(\Gamma_\xi) = \{v \in V(\Gamma_\xi) - V^S(\Gamma_\xi) \mid |S_v| = a, |E_v| = b\}.$$

We have a disjoint union

$$V(\Gamma_\xi) = V^S(\Gamma_\xi) \cup V^{0,1}(\Gamma_\xi) \cup V^{0,2}(\Gamma_\xi) \cup V^{1,1}(\Gamma_\xi).$$

We also adopt the convention $V_c^{a,b}(\Gamma_\xi) = V_c(\Gamma_\xi) \cap V^{a,b}(\Gamma_\xi)$.

Let

$$(2.7) \quad \Delta_{g,\gamma,\mathbf{d}} = \{\Gamma_\xi \mid \xi \in \mathcal{W}^T\} / \cong, \quad \Delta = \coprod \Delta_{g,\gamma,\mathbf{d}},$$

where \cong is the equivalence induced by isomorphisms of graphs preserving datum in Definitions 2.5 and 2.6, including the partition $V(\Gamma) = V_0(\Gamma) \cup V_1(\Gamma) \cup V_\infty(\Gamma)$, etc.. The disjoint union is taken over all possible (g, γ, \mathbf{d}) .

2.3. Decompositions of torus fixed MSP-fields. We continue to abbreviate $\mathcal{W}_{g,\gamma,\mathbf{d}}$ by \mathcal{W} , as in (2.3). Let $\xi \in \mathcal{W}^T$. We will characterize the structure of \mathcal{C}_a and $\xi|_{\mathcal{C}_a}$ for each $a \in V^S(\Gamma_\xi) \cup E(\Gamma_\xi)$. First, by allowing rational weights, we can assume the exponent $\varrho = 1$ in (2.4). Then that ξ is T -fixed can be summarized in that there are representation h_t and linearizations τ_t and τ'_t as in (2.4) so that under their induced actions, together with \mathbf{L}_k the dimension one weight k T -representation,

$$\varphi \in H^0(\mathcal{L}^{\oplus 5})^T, \quad \rho \in H^0(\mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}}^{\log})^T, \quad (\nu_1, \nu_2) \in H^0(\mathcal{L} \otimes \mathcal{N} \otimes \mathbf{L}_1 \oplus \mathcal{N})^T$$

are T -invariant. As a direct consequence, we have the following remark.

Remark 2.7. When $p \in \mathcal{C}_0$, then as $\varphi|_p$ and $\nu_2|_p$ are nonzero, we have $\mathcal{L}|_p \cong \mathcal{N}|_p \cong \mathbf{L}_0$. When $p \in \mathcal{C}_1$, then $\nu_1|_p$ and $\nu_2|_p$ are nonzero and hence $\mathcal{L} \otimes \mathcal{N} \otimes \mathbf{L}_1|_p \cong \mathcal{N}|_p \cong \mathbf{L}_0$. When $p \in \mathcal{C}_\infty$, then $\rho|_p$ and $\nu_1|_p$ are nonzero and thus $\mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}}^{\log}|_p \cong \mathcal{L} \otimes \mathcal{N} \otimes \mathbf{L}_1|_p \cong \mathbf{L}_0$.

Example 2.8 (Stable maps with p -fields). A stable MSP-field $\xi \in \mathcal{W}$ having $\nu_1 = 0$ will have $\mathcal{N} \cong \mathcal{O}_{\mathcal{C}}$, $\nu_2 = 1$, and then ξ is a pair of a stable map $f = [\varphi] : \Sigma^{\mathcal{C}} \subset \mathcal{C} \rightarrow \mathbb{P}^4$ and a p -field $\rho \in H^0(f^*\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_{\mathcal{C}})$.

Example 2.9 (5-spin curves with five p -fields). A stable MSP-field $\xi \in \mathcal{W}$ having $\nu_2 = 0$ will have $\mathcal{N} \cong \mathcal{L}^{\vee}$, $\nu_1 = 1$ and then ξ is a pair of a 5-spin curve $(\Sigma^{\mathcal{C}}, \mathcal{C}, \rho : \mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}^{\log})$ and five p -fields $\varphi \in H^0(\mathcal{L})^{\oplus 5}$, with $\varphi|_{\Sigma_{(1,\varphi)}^{\mathcal{C}}} = 0$.

Lemma 2.10. Let $\xi \in \mathcal{W}^T$, and suppose $E_{0\infty}(\Gamma_\xi) = \emptyset$.

- (a) For $v \in V_0^S(\Gamma_\xi)$, the sections $(\varphi_1, \dots, \varphi_5)|_{\mathcal{C}_v}$ define a morphism $f_v : \mathcal{C}_v \rightarrow \mathbb{P}^4$; together with the marking $\Sigma^{\mathcal{C}_v} = \Sigma_{\text{inn}}^{\mathcal{C}_v} \cup \Sigma_{\text{out}}^{\mathcal{C}_v}$ and $\rho_v = \rho|_{\mathcal{C}_v}$, they form an $E_v \cup S_v$ -pointed, genus g_v , degree d_v stable map to \mathbb{P}^4 with p -field.
- (b) For $v \in V_1^S(\Gamma)$, $\Sigma^{\mathcal{C}_v} \subset \mathcal{C}_v$ is an $E_v \cup S_v$ -pointed, genus g_v stable curve.
- (c) For $v \in V_\infty^S(\Gamma)$, let $\gamma_v = \{\gamma_i : i \in S_v\} \cup \{\gamma_{(e,v)} : e \in E_v\}$. Suppose all γ_i and $\gamma_{(e,v)}$ in γ_v lie in $\mu_5 - \{1\}$, then $(\Sigma^{\mathcal{C}_v}, \mathcal{C}_v)$ with $(\mathcal{L}, \rho, \varphi)|_{\mathcal{C}_v}$ is a 5-spin $E_v \cup S_v$ -pointed twisted curve with five p -fields.

Proof. The proof is straightforward, and will be omitted. Note that $E_{0\infty}(\Gamma_\xi) = \emptyset$ guarantees that $\varphi|_{y_{(e,v)}} = 0$ in case (c) and $\rho|_{y_{(e,v)}} = 0$ in case (a). \square

We now move to $e \in E_0(\Gamma_\xi)$. By Lemma 2.4 we know $\mathcal{C}_e \cong \mathbb{P}^1$. Let $v \in V_0(\Gamma_\xi)$ and $v' \in V_1(\Gamma_\xi)$ with (e, v) and $(e, v') \in F(\Gamma)$. Let $q = \mathcal{C}_e \cap \mathcal{C}_v$ and $q' = \mathcal{C}_e \cap \mathcal{C}_{v'}$. We give \mathcal{C}_e the homogeneous coordinate $[x, y]$ so that $[1, 0] = q$ and $[0, 1] = q'$. Recall that $d_e = d_{0e} - d_{\infty e}$.

Lemma 2.11. *Let the notations be as stated. Then $\mathcal{N}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathbb{P}^1}$, $\mathcal{L}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathbb{P}^1}(d_e)$, and for some $(c_1, \dots, c_5) \in \mathbb{C}^5 - 0$ and $c \in \mathbb{C}^\times$,*

$$(\varphi_1, \dots, \varphi_5; \rho; \nu_1, \nu_2)|_{\mathcal{C}_e} = (c_1 x^{d_e}, \dots, c_5 x^{d_e}; 0; cy^{d_e}, 1).$$

Proof. Because $\deg \mathcal{L}|_{\mathcal{C}_e} = d_e$, $\mathcal{L}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathbb{P}^1}(d_e)$. Since $q \in \mathcal{C}_0$, the linearization $\tau_t|_q = \text{id}$. Thus the induced T -linearization on $H^0(\mathcal{L}|_{\mathcal{C}_e}) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(d_e))$ is via $t \cdot x^{d_e-i} y^i = t^{ik} x^{d_e-i} y^i$, for some $k \neq 0$, and $H^0(\mathcal{O}_{\mathbb{P}^1}(d_e))^T = \mathbb{C} \cdot x^{d_e}$. Since $\varphi_i|_{\mathcal{C}_e}$ are T -invariant, and $\nu_1|_{\mathcal{C}_e}$ is T -equivariant (of certain weight) and non-vanishing at q' , there are $c, c_i \in \mathbb{C}$ so that $\phi_i|_{\mathcal{C}_e} = c_i x^{d_e}$ and $\nu_1|_{\mathcal{C}_e} = cy^{d_e}$. As $\nu_1(q') \neq 0$, $c \neq 0$; as $\varphi(q) \neq 0$, $(c_1, \dots, c_5) \neq 0$. This proves the lemma. \square

We next consider $e \in E_\infty(\Gamma_\xi)$. By Lemma 2.11 we know that \mathcal{C}_e is a smooth rational curve. It is isomorphic to \mathbb{P}^1 when $d_e \in \mathbb{Z}$, and isomorphic to $\mathbb{P}(5, 1)$ and having $\mathcal{C}_e \cap \mathcal{C}_\infty$ its only stacky point when $d_e \notin \mathbb{Z}$. Since $\nu_1|_{\mathcal{C}_e}$ is nowhere vanishing, any linearization on $\mathcal{L}|_{\mathcal{C}_e}$ will induce a linearization on $\mathcal{N}|_{\mathcal{C}_e}$ keeping $\nu_1|_{\mathcal{C}_e}$ a T -invariant section of $\mathcal{L} \otimes \mathcal{N}|_{\mathcal{C}_e} \otimes \mathbf{L}_1$.

Convention on $\mathbb{P}(5, 1)$. Let $(\mathcal{X}, q, q') = (\mathbb{P}(5, 1), [1, 0], [0, 1])$, with a homogeneous coordinates $[\tilde{x}, \tilde{y}]$, where $q = [1, 0]$ is its only stacky point. And $x = \tilde{x}$ and $y = \tilde{y}^5$ descend to its coarse moduli $X = \mathbb{P}^1$ and form a corresponding homogeneous coordinates. For $c \in \frac{1}{5}\mathbb{Z}$, we agree $\mathcal{O}_{\mathcal{X}}(c) \cong \mathcal{O}_{\mathcal{X}}(5c, q)$ and $\deg \mathcal{O}_{\mathcal{X}}(q) = 1/5$. Thus $H^0(\mathcal{O}_{\mathcal{X}}(c)) \cong H^0(\mathcal{O}_X([c]))$. In the notation of \mathcal{M}_m before (2.18), near q $\mathcal{O}_{\mathcal{X}}(c)$ is the module $\tilde{y}^{-m}\mathbb{C}[\tilde{y}]$, where $m = 5c - 5[c]$.

Let $v \in V_\infty(\Gamma_\xi)$ and $v' \in V_1(\Gamma_\xi)$ be vertices so that (e, v) and $(e, v') \in F(\Gamma_\xi)$, and let $q = \mathcal{C}_e \cap \mathcal{C}_v$ and $q' = \mathcal{C}_e \cap \mathcal{C}_{v'}$. We endow \mathcal{C}_e a homogeneous coordinate $[x, y]$ (resp. $[\tilde{x}, \tilde{y}]$) so that $q = [1, 0]$ and $q' = [0, 1]$ when $\mathcal{C}_e \cong \mathbb{P}^1$ (resp. $\mathbb{P}(5, 1)$), following the convention just stated.

We set $\delta = -1$ when $\dim \mathcal{C}_v = 0$, $S_v = \emptyset$ and $|E_v| = 1$ (equivalently $v \in V_{\infty}^{0,1}(\Gamma_{\xi})$), and $\delta = 0$ otherwise, and set $\delta' = -1$ when $v' \in V_1^{0,1}(\Gamma_{\xi})$, and $\delta' = 0$ otherwise. Note that when $d_e = d_{0e} - d_{\infty e} \notin \mathbb{Z}$, $\delta = 0$.

Lemma 2.12. *Let $e \in E_{\infty}(\Gamma_{\xi})$ be as stated. Then $\varphi|_{\mathcal{C}_e} = 0$, $\mathcal{N}|_{\mathcal{C}_e} \cong \mathcal{L}^{\vee}|_{\mathcal{C}_e}$, and $\nu_1|_{\mathcal{C}_e}$ is never zero. Furthermore, $\omega_{\mathcal{C}_e}^{\log}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathcal{C}_e}(\delta')$, $\mathcal{L}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathcal{C}_e}(d_e)$, and $(\rho, \nu_2)|_{\mathcal{C}_e} = (cx^{-5d_e+\delta+\delta'}, c'y^{\lfloor -d_e \rfloor})$ for some $c, c' \in \mathbb{C}^{\times}$.*

Proof. The part $\varphi|_{\mathcal{C}_e} = 0$, etc., is merely restating the known facts. We now consider the case $\mathcal{C}_e \cong \mathbb{P}^1$. Again, $\omega_{\mathcal{C}_e}^{\log}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathbb{P}^1}(\delta + \delta')$ and $\mathcal{L}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathbb{P}^1}(d_e)$ follow from the definition. By Remark 2.7, we have $\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}_e}^{\log}|_q \cong \mathbf{L}_0 \cong \mathcal{N}|_{q'}$ as T -representations. Thus

$$H^0(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}_e}^{\log}|_{\mathcal{C}_e})^T = \mathbb{C} \cdot x^{-5d_e+\delta+\delta'} \quad \text{and} \quad H^0(\mathcal{N}|_{\mathcal{C}_e})^T = \mathbb{C} \cdot y^{-d_e}.$$

This proves the furthermore part of the lemma in this case.

We now prove the furthermore part when $\mathcal{C}_e \cong \mathbb{P}(5, 1)$. In this case, we know $\delta = 0$, $\omega_{\mathcal{C}_e}^{\log}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathbb{P}(5,1)}(\delta')$ which follows from the definition of $V^{a,b}(\Gamma)$, and $\mathcal{L}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathbb{P}(5,1)}(d_e)$ by its degree. Like before, using that as T -representations $\mathcal{N}|_{q'} \cong \mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}_e}^{\log}|_q \cong \mathbf{L}_0$, we conclude that (see convention,)

$$H^0(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}_e}^{\log}|_{\mathcal{C}_e})^T = \mathbb{C} \cdot x^{-5d_e+\delta'} \quad \text{and} \quad H^0(\mathcal{N}|_{\mathcal{C}_e})^T = \mathbb{C} \cdot y^{\lfloor -d_e \rfloor}.$$

This proves the lemma. \square

Lastly, we consider $e \in E_{0\infty}(\Gamma_{\xi})$. Let (e, v) and $(e, v') \in F(\Gamma_{\xi})$ with $v \in V_{\infty}(\Gamma_{\xi})$ and $v' \in V_0(\Gamma_{\xi})$. Let $q = \mathcal{C}_e \cap \mathcal{C}_v$ and $q' = \mathcal{C}_e \cap \mathcal{C}_{v'}$.

Lemma 2.13. *Let $e \in E_{0\infty}(\Gamma_{\xi})$ be as stated. Then $\mathcal{C}_e \cong \mathbb{P}^1$, q is a node of \mathcal{C} , $\mathcal{L}|_{\mathcal{C}_e} \cong \omega_{\mathcal{C}_e}^{\log}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathcal{C}_e}$ and $\mathcal{N}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathcal{C}_e}(d_{\infty e})$. Furthermore, for homogeneous coordinates $[x, y]$ of \mathcal{C}_e so that $q = [1, 0]$ and $q' = [0, 1]$, there are constants c 's so that $\nu_1|_{\mathcal{C}_e} = cx^{d_{\infty e}}$, $\nu_2|_{\mathcal{C}_e} = c'y^{d_{\infty e}}$, $\varphi|_{\mathcal{C}_e} = (c_1, \dots, c_5)$ and $\rho|_{\mathcal{C}_e} = c''$.*

Proof. Since $\varphi(q') \neq 0$, we have $\deg \mathcal{L}|_{\mathcal{C}_e} \geq 0$. Since $\rho(q) \neq 0$, we have $\deg \mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}_e}^{\log}|_{\mathcal{C}_e} \geq 0$. Furthermore, since $\deg \omega_{\mathcal{C}_e}^{\log}|_{\mathcal{C}_e} \leq 0$, we have $\deg \mathcal{L}|_{\mathcal{C}_e} = \deg \omega_{\mathcal{C}_e}^{\log}|_{\mathcal{C}_e} = 0$. Hence $\mathcal{C}_e \cong \mathbb{P}^1$ and q is either a node or a marking. We claim that q is not a marking. Suppose not, since $\rho(q) \neq 0$ and q is a scheme, q must be labeled by $(1, \varphi)$. On the other hand, as $\deg \mathcal{L}|_{\mathcal{C}_e} = 0$ and $\varphi(q') \neq 0$, we must have $\varphi(q) \neq 0$, which is impossible. This proves that q is a node of \mathcal{C} . The remaining part of the lemma is straightforward. \square

We introduce the notion of T -balanced nodes.

Definition 2.14. *Let \mathcal{C} be a T -twisted curve and q be a node of \mathcal{C} . Let $\hat{\mathcal{C}}_1$ and $\hat{\mathcal{C}}_2$ be the two branches of the formal completion of \mathcal{C} along q . We call q T -balanced if $T_q \hat{\mathcal{C}}_1 \otimes T_q \hat{\mathcal{C}}_2 \cong \mathbf{L}_0$ as T -representations.*

Let $v \in V_1(\Gamma_\xi)$ be an unstable vertex with two distinct (e, v) and $(e', v) \in F(\Gamma_\xi)$, and let $q_v = \mathcal{C}_e \cap \mathcal{C}_{e'}$ be the node of \mathcal{C} associated with v .

Lemma 2.15. *Let $v \in V_1(\Gamma_\xi)$ be an unstable vertex, etc., as stated. Then q_v is T -balanced if and only if $d_e + d_{e'} = 0$, and $(\mathcal{C}_e \cup \mathcal{C}_{e'}) \cap \mathcal{C}_\infty$ is a node or a marking of \mathcal{C} .*

Proof. We first consider the case where $e \in E_\infty(\Gamma_\xi)$ and $e' \in E_0(\Gamma_\xi)$. Let $q = \mathcal{C}_e \cap \mathcal{C}_\infty$ and $q' = \mathcal{C}_{e'} \cap \mathcal{C}_0$. By Remark 2.7,

$$(2.8) \quad \mathcal{L}|_{q'} \cong \mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}}^{\log}|_q \cong \mathbf{L}_0.$$

Let $\delta = 0$ when q is a node or a marking of \mathcal{C} , and $= -1$ otherwise. Then since q_v is a node, $\omega_{\mathcal{C}}^{\log}|_{q_v} \cong \mathbf{L}_0$, and thus $\mathcal{L}^{\vee \otimes 5}|_{q_v} \cong \mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}}^{\log}|_{q_v}$ as T -representations. Since q_v is a scheme point, and since $\mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}}^{\log}|_{\mathcal{C}_e}$ is a pullback of an invertible sheaf on the coarse moduli of \mathcal{C}_e , combined with (2.8), we conclude that q_v is a T -balanced node if and only if

$$\deg \mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}}^{\log}|_{\mathcal{C}_e} + \deg \mathcal{L}^{\vee \otimes 5}|_{\mathcal{C}_{e'}} = 0,$$

which is $-5d_e + \delta - 5d_{e'} = 0$. We claim that then $\delta = 0$ and $d_e + d_{e'} = 0$. Indeed, if $\delta = -1$, then \mathcal{C}_e is a scheme and $d_e \in \mathbb{Z}$ (and $d_{e'} \in \mathbb{Z}$), impossible. Thus $\delta = 0$ and the lemma follows.

The other cases where both e and e' lie in $E_0(\Gamma_\xi)$ or in $E_\infty(\Gamma_\xi)$ can be ruled out by a similar argument. This proves the lemma. \square

Let

$$(2.9) \quad N(\Gamma_\xi) = V^{0,2}(\Gamma_\xi) \cup \{(e, v) \in F(\Gamma_\xi) \mid v \in V^S(\Gamma_\xi)\}.$$

Note that every $a \in N(\Gamma_\xi)$ has its associated node q_a of \mathcal{C} .

Definition 2.16. *We call $a \in N(\Gamma_\xi)$ T -balanced if the associated node q_a is a T -balanced node in \mathcal{C} . Let $N(\Gamma_\xi)^{\text{un}} \subset N(\Gamma_\xi)$ be the subset of T -unbalanced elements.*

Proposition 2.17. *Let $\xi \in \mathcal{W}^T$ with decorated graph Γ_ξ . Then q_a , $a \in N(\Gamma_\xi)$, is T -balanced if and only if $a \in V_1^{0,2}(\Gamma_\xi)$ and satisfies the condition in Lemma 2.15. Furthermore, the set of T -unbalanced nodes of \mathcal{C} is $\{q_a \mid a \in N(\Gamma_\xi)^{\text{un}}\}$.*

Proof. Let q_a be a node indexed by $a \in N(\Gamma_\xi)$. Then a repetition of the proof of Lemma 2.15 shows that q_a is T -balanced only if $a \in V_1^{0,2}(\Gamma_\xi)$, and satisfies the criterion in Lemma 2.15.

To complete the proof, we need to show that any node q of \mathcal{C} not listed in $\{q_a \mid a \in N(\Gamma_\xi)\}$, must be T -balanced. Indeed, such q must be a node of one of \mathcal{C}_0 , \mathcal{C}_1 and \mathcal{C}_∞ . Since T acts trivially on these curves, q must be a balanced node. This proves the proposition. \square

Although a T -balanced $a \in N(\Gamma_\xi)$ is characterized by q_a being T -balanced, by Lemma 2.15 it can also be characterized by the information of the (decorated) graph Γ_ξ . Thus for any $\Gamma \in \Delta$, we can talk about $N(\Gamma)^{\text{un}} \subset N(\Gamma)$.

Definition 2.18. A decorated graph Γ is called a flat graph if $N(\Gamma)^{\text{un}} = N(\Gamma)$.

2.4. Γ -framed MSP fields. We recall the convention of decomposing a family of nodal curves along its nodal locus. We continue to denote $\mathcal{W} = \mathcal{W}_{g,\gamma,\mathbf{d}}$.

Definition 2.19. Let S be a scheme and let \mathcal{C}_S be a flat S -family of twisted curves. We say that \mathcal{C}_S can be decomposed along nodes $\mathcal{Q}_S \subset \mathcal{C}_S$ if \mathcal{Q}_S is a closed substack of \mathcal{C}_S that is a gerbe over S and homeomorphic to S , so that there are a flat S -family of twisted curves $\tilde{\mathcal{C}}_S$, an S -morphism $\tilde{\mathcal{C}}_S \rightarrow \mathcal{C}_S$, and two disjoint closed S -embeddings $h_1, h_2 : \mathcal{Q}_S \rightarrow \tilde{\mathcal{C}}_S$ so that

$$\text{hom}_S(\mathcal{C}_S, \cdot) = \{f \in \text{hom}_S(\tilde{\mathcal{C}}_S, \cdot) \mid h_1 \circ f \approx h_2 \circ f\}.$$

(See [AGV, A.1] for precise formulation of \approx .) If \mathcal{C}_S can be decomposed along \mathcal{Q}_S , we call \mathcal{Q}_S an S -family of nodes.

If \mathcal{C} is a twisted curve with only one node $q \in \mathcal{C}$, then the decomposition of \mathcal{C} along q is the normalization $\tilde{\mathcal{C}}$ of \mathcal{C} , and $h_1, h_2 : q \rightarrow \tilde{\mathcal{C}}$ are the two preimages of q via the projection $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$. Note that we can decompose along several disjoint S -families of nodes by applying this procedure repeatedly to each S -family of nodes.

We now assume that \mathcal{C}_S is an S -family of T -curves, meaning that T acts on \mathcal{C}_S and commutes with the trivial T -action on S . We have the following well-known existence of decomposition along families of nodes.

Lemma 2.20. Let $s \in S$ be a closed point in a DM stack, let \mathcal{C}_S be a flat S -family of nodal twisted T -curves, and let $q \in \mathcal{C}_s$ be a T -unbalanced node of \mathcal{C}_s . Then there is an étale neighborhood $\tilde{s} \in \tilde{S} \rightarrow S$, $\tilde{s} \mapsto s$, so that we can extend $\tilde{q} = q \times_{\tilde{s}} \tilde{s} \in \mathcal{C}_s|_{\tilde{s}}$ to an \tilde{S} -family of nodes $\mathcal{Q}_{\tilde{S}} \subset \mathcal{C}_S|_{\tilde{S}}$.

A simple observation shows that such decomposition may not exist for T -balanced nodes. Thus we need to modify the graph Γ_ξ in case $N(\Gamma_\xi)$ contains T -balanced node in order to be able to decompose families along nodes characterized by the graph. To achieve this, we construct a flattening of Γ_ξ which eliminates T -balanced nodes from the graph.

Construction 2.21. For each T -balanced $v \in N(\Gamma_\xi)$, which necessarily is an unstable vertex in $V_1(\Gamma_\xi)$, we eliminate the vertex v from Γ_ξ , replace the two edges $e \in E_\infty(\Gamma_\xi)$ and $e' \in E_0(\Gamma_\xi)$ incident to v by a single edge \tilde{e} incident to the other two vertices that are incident to e or e' , and demand that \tilde{e} lies in $E_{0\infty}$. For the decorations, we agree $\tilde{g}(\tilde{e}) = 0$ and $(d_{0\tilde{e}}, d_{\infty\tilde{e}}) = (d_{\infty e}, d_{\infty e'})$, while keeping the rest unchanged. The resulting decorated graph is called the flattening of Γ_ξ at v .

Let Γ_ξ^{fl} be the graph after applying this to all T -balanced v in $N(\Gamma_\xi)$. We call it the flattening of Γ_ξ . As the result, $N(\Gamma_\xi^{\text{fl}}) = N(\Gamma_\xi)^{\text{un}}$, or equivalently Γ_ξ^{fl} is a flat graph.

Definition 2.22 (Standard decomposition). *For any $\xi \in \mathcal{W}^T$ with the domain curve \mathcal{C} , we can decompose \mathcal{C} along its T -unbalanced nodes q_a , $a \in N(\Gamma_\xi)^{\text{un}} = N(\Gamma_\xi^{\text{fl}})$, to obtain connected subcurves \mathcal{C}_b indexed by*

$$b \in \Xi(\Gamma_\xi^{\text{fl}}) := V^S(\Gamma_\xi^{\text{fl}}) \cup E(\Gamma_\xi^{\text{fl}}).$$

Definition 2.23. *Given $\Gamma \in \Delta$ so that $\Gamma = \Gamma^{\text{fl}}$, and given a scheme S , a Γ -framed S -family in \mathcal{W}^T consists of a pair (ξ_S, ϵ_S) , where*

$$(2.10) \quad \xi_S \in \mathcal{W}^T(S) \quad \text{and} \quad \epsilon_S = \{\epsilon_s : \Gamma \cong \Gamma_{\xi_s}^{\text{fl}} \mid s \in S \text{ a closed point}\},$$

such that if for each closed point $s \in S$ we let $q_{s,a} = q_{s,\epsilon_s(a)}$ for $a \in N(\Gamma)$ and $\mathcal{C}_{s,b} = \mathcal{C}_{s,\epsilon_s(b)}$ for $b \in \Xi(\Gamma)$ be the result of the standard decomposition of $\xi_S \times_S s = (\mathcal{C}_s, \dots)$, then the unions $\coprod_{s \in S} q_{s,a}$ and $\coprod_{s \in S} \mathcal{C}_{s,b}$ are closed substacks of the domain curve \mathcal{C}_S of ξ_S , and are flat over S_{red} .

Applying Lemma 2.20, we see that given a connected scheme and any Γ -framed S -family in \mathcal{W}^T with \mathcal{C}_S its total space of base curves, we have an S -family of nodes $\mathcal{Q}_{S,a} \subset \mathcal{C}_S$ indexed by $a \in N(\Gamma)$ so that they decompose \mathcal{C}_S into S -flat families of connected subcurves $\mathcal{C}_{S,b}$ indexed by $b \in \Xi(\Gamma)$. (Here $\Gamma = \Gamma^{\text{fl}}$ automatically when we speak of Γ -framed families.)

Define

$$\Delta_{g,\gamma,\mathbf{d}}^{\text{fl}} := \{\Gamma^{\text{fl}} \mid \Gamma \in \Delta_{g,\gamma,\mathbf{d}}\} / \cong, \quad \Delta^{\text{fl}} = \bigcup \Delta_{g,\gamma,\mathbf{d}}^{\text{fl}}.$$

Here \cong is as that in (2.7). Let $\Gamma \in \Delta^{\text{fl}}$, and let \mathcal{W}_Γ be the groupoid of Γ -framed families in \mathcal{W}^T with obviously defined arrows. Then \mathcal{W}_Γ is a DM-stack, finite over \mathcal{W}^T via the forgetful morphism $\mathcal{W}_\Gamma \rightarrow \mathcal{W}^T$.

Proposition 2.24. *Let $\mathcal{W}_{(\Gamma)}$ be the image of the forgetful morphism $\iota_\Gamma : \mathcal{W}_\Gamma \rightarrow \mathcal{W}^T$. Then $\mathcal{W}_{(\Gamma)}$ is an open and closed substack of \mathcal{W}^T . Let ι_Γ be factored as*

$$(2.11) \quad \iota_\Gamma : \mathcal{W}_\Gamma \xrightarrow{\imath_\Gamma} \mathcal{W}_{(\Gamma)} \xrightarrow{\jmath_\Gamma} \mathcal{W}^T.$$

Then \imath_Γ is an $\text{Aut}(\Gamma)$ -torsor.

Proof. That $\mathcal{W}_{(\Gamma)}$ is open and closed follows from lemma 2.20 and the definition of Γ -framed MSP fields, the other parts are straightforward to prove, and will be omitted. \square

We end this subsection with the notion of Γ -framed curve $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N})$. Recall that given a flat Γ and a $(\xi, \epsilon) \in \mathcal{W}_\Gamma$, where $\xi = (\mathcal{C}, \dots)$, etc., we not only have an identification of the T -unbalanced nodes of \mathcal{C} with $N(\Gamma)$, but also have a tautological identification of branches of these T -unbalanced nodes: Let $q \in \mathcal{C}$ be a T -unbalanced node. Then we have

- (I) when q is identified with an $(e, v) \in N(\Gamma)$, $v \in V^S(\Gamma)$, then the two branches of \mathcal{C} along q are labeled by v and e ;
- (II) when q is identified with a $v \in V^U(\Gamma)$, then the two branches of \mathcal{C} along q are labeled by (e_1, v) and $(e_2, v) \in F(\Gamma)$.

Definition 2.25. A Γ -framed curve consists of T -equivariant $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N})^1$ such that

- (1) $\Sigma^{\mathcal{C}}$ is labeled by legs in Γ ;
- (2) T -unbalanced nodes of \mathcal{C} are labeled by $N(\Gamma)$;
- (3) branches of T -unbalanced nodes of \mathcal{C} are labeled according to rule I and II above.

Because the conditions in Definition 2.25 are open, we can speak of flat family of Γ -framed curves. Let \mathcal{D}_{Γ} be the stack of flat families of Γ -framed curves, where arrows are T -equivariant arrows in \mathcal{D} that preserve the labeling in Definition 2.25. Here \mathcal{D} is the stack of data $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N})$, where $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ are pointed twisted curves, and \mathcal{L} and \mathcal{N} are invertible sheaves on \mathcal{C} . Clearly, \mathcal{D}_{Γ} is a smooth Artin stack, and admits a forgetful morphism $\mathcal{W}_{\Gamma} \rightarrow \mathcal{D}_{\Gamma}$.

2.5. Decomposition via decoupling. We introduce a *decoupling* operation on flat graphs Γ . We use decoupling of the graph Γ to decompose \mathcal{W}_{Γ} as a product of simpler moduli spaces. Let Γ be a flat decorated graph, and let

$$N(\Gamma)^* = V_1^{0,2}(\Gamma) \cup V_{\infty}^{0,2} \cup \{(e, v) \in F(\Gamma) \mid v \in V_1^S(\Gamma) \cup V_{\infty}^S(\Gamma), e \in E_{\infty} \cup E_0\}.$$

It is subset of $N(\Gamma)$.

Definition 2.26. Let $\Gamma \in \Delta^{\text{fl}}$ be a flat graph, and let $a \in N(\Gamma)^*$. Let $a = (e, v)$ and define the decoupling of Γ along a to be a new graph $\bar{\Gamma}$ resulting from Γ by the following modifications:

- (1) add a new vertex \bar{v} to Γ and replace the flag (e, v) by (e, \bar{v}) ;
- (2) add a leg l to v and a leg \bar{l} to \bar{v} , decorated via:
 - when $v \in V_1(\Gamma)$, we decorate them by $(1, \rho)$;
 - when $v \in V_{\infty}(\Gamma)$ and $d_{e,0} \notin \mathbb{Z}$, l is decorated by $\gamma_{(e,v)}$ and \bar{l} by $\gamma_{(e,v)}^{-1}$;
 - when $v \in V_{\infty}(\Gamma)$ and $d_{e,0} \in \mathbb{Z}$, we decorat l and \bar{l} by $(1, \varphi)$.

Note that in case $a \in V^{0,2}(\Gamma)$, $a = (e, v)$ or (e', v) . However, the decoupling using (e, v) and that using (e', v) are canonically isomorphic. Note that the resulting graph is possibly not connected.

We apply decoupling repeatedly to all $a \in N(\Gamma)^*$. Let Γ^{de} be the resulting graph and denote by $(\Gamma^{\text{de}})_{\text{conn}}$ the set of connected components of Γ^{de} .

For each $A \in (\Gamma^{\text{de}})_{\text{conn}}$, we form the moduli \mathcal{W}_A of A -framed MSP fields. Let $\mathcal{W}_{\Gamma^{\text{de}}}$ be the moduli of Γ^{de} -framed MSP fields. Note that though Γ^{de} could be disconnected, the construction \mathcal{W}_{Γ} applies and $\mathcal{W}_{\Gamma^{\text{de}}}$ is well-defined, resulting a stack naturally isomorphic to the product of \mathcal{W}_A for all $A \in (\Gamma^{\text{de}})_{\text{conn}}$.

¹By T -equivariant we mean that $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ comes with a T -action, and both \mathcal{L} and \mathcal{N} are T -linearized.

Lemma 2.27. *For each $A \in (\Gamma^{\text{de}})_{\text{conn}}$, the decoupling defines a natural morphism $\Phi_A : \mathcal{W}_\Gamma \longrightarrow \mathcal{W}_A$, whose products defines a morphism*

$$\Phi = \prod \Phi_A : \mathcal{W}_\Gamma \longrightarrow \mathcal{W}_{\Gamma^{\text{de}}} = \prod_{A \in (\Gamma^{\text{de}})_{\text{conn}}} \mathcal{W}_A.$$

Proof. Let $a \in N(\Gamma)^*$ and let Γ' be the decoupling of Γ along a . Let $\mathcal{W}_{\Gamma'}$ be the moduli of Γ' -framed MSP fields. We show that there is a natural morphism

$$(2.12) \quad \phi : \mathcal{W}_\Gamma \longrightarrow \mathcal{W}_{\Gamma'}.$$

Let $\xi_T \in \mathcal{W}_\Gamma(T)$ be any family over T , say given by $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$. Because v is stable, by the definition of Γ -framed MSP fields, the flag (e, v) corresponds to a T family of nodes $\mathbf{n}_{(e,v)} \subset \mathcal{C}$. Let \mathcal{C}' be the partial normalization of \mathcal{C} along $\mathbf{n}_{(e,v)}$, namely, \mathcal{C}' is obtained from \mathcal{C} with the two branches of the node $\mathbf{n}_{(e,v)}$ separated.

Then let $\Sigma^{\mathcal{C}'} = \Sigma^{\mathcal{C}} \times_{\mathcal{C}} \mathcal{C}'$, and let $(\mathcal{L}', \mathcal{N}', \varphi', \rho', \nu')$ be the pullback of $(\mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$ via the tautological projection $\mathcal{C}' \rightarrow \mathcal{C}$. It is direct to check that $(\mathcal{C}', \Sigma^{\mathcal{C}'}, \mathcal{L}', \mathcal{N}', \varphi', \rho', \nu')$ is a T -family of Γ' -framed MSP fields. As this construction is canonical, it commutes with base change. Thus it defines a morphism $\phi : \mathcal{W}_\Gamma \rightarrow \mathcal{W}_{\Gamma'}$, as desired. Since Γ^{de} is derived by repeated applying decoupling to selected (e, v) , by induction, we obtain the desired Φ . \square

Our next step is to show that Φ is an isomorphism. To this end, we introduce more notations. For convenience, for a vertex $v \in V_\infty(\Gamma)$ with $\gamma_v = \{\zeta_5^{a_1}, \dots, \zeta_5^{a_c}\}$, we abbreviate $\gamma_v = (0^{e_0} \dots 4^{e_4})$, where e_i is the number of appearances of i in $\{a_1, \dots, a_c\}$. We call a vertex $v \in V_\infty^S(\Gamma)$ exceptional if $g_v = 0$ and $\gamma_v = (1^{2+k}4)$ or $(1^{1+k}23)$ for some $k \geq 0$.

Definition 2.28. *We call a $v \in V_\infty(\Gamma)$ regular if the followings hold:*

- (1) *In case v is stable, then either v is exceptional, or for every $a \in S_v$ and $e \in E_v$ we have γ_a and $\gamma_{(e,v)} \in \{\zeta_5, \zeta_5^2\}$.*
- (2) *In case v is unstable and \mathcal{C}_v is a scheme point, then \mathcal{C}_v is neither a node nor a marking of $\Sigma^{\mathcal{C}} \subset \mathcal{C}$.*

We call Γ regular if it is flat, and such that every $v \in V_\infty(\Gamma)$ is regular; otherwise we call it irregular. We denote by Δ^{reg} the collection of regular graphs in Δ^{fl} .

Let $v \in V^S(\Gamma)$. We denote by $|v|$ the one vertex graph consisting of the vertex v and legs labeled by $E_v \cup S_v$. Thus $|v|$ has inherited old legs corresponding to S_v from Γ as well as new legs from E_v . For decoration, except legs, v in $|v|$ is decorated by the same data as v in Γ , for instance $|v|$ lies in $V_a(|v|)$ when $v \in V_a(\Gamma)$. For legs, old legs inherited from Γ have the same decorations as those in Γ . For new legs, when $v \in V_0(\Gamma)$, all new legs in $|v|$ are decorated by $(1, \rho)$; when $v \in V_1(\Gamma)$, new legs in $|v|$ are also decorated by $(1, \rho)$; when $v \in V_\infty(\Gamma)$, the new leg in $|v|$ indexed by $a \in E_v$ is decorated by

γ_a . (cf. γ_a is defined in the paragraph before (2.6).) Let \mathcal{W}_v be the moduli space of $|v|$ -framed MSP fields.

Proposition 2.29. *When $v \in V_0^S(\Gamma)$ or $v \in V_1^S(\Gamma)$, the correspondences given in Lemma 2.10 define canonical isomorphisms*

$$\mathcal{W}_v \cong \overline{\mathcal{M}}_{g_v, E_v \cup S_v}(\mathbb{P}^4, d_v)^p \quad \text{or} \quad \mathcal{W}_v \cong \overline{\mathcal{M}}_{g_v, E_v \cup S_v},$$

respectively. When $v \in V_\infty^S(\Gamma)$ is regular, we have canonical isomorphism

$$\mathcal{W}_v \cong \overline{\mathcal{M}}_{g_v, \gamma_v}^{1/5, 5p}.$$

Here by $E_v \cup S_v$ -pointed we mean that the markings are labeled by elements in $E_v \cup S_v$.

Proof. This is the family version of Lemma 2.10. We omit the details here. \square

Let $e \in E_0(\Gamma)$, and let \mathcal{W}_e be the stack parameterizing families of

$$(2.13) \quad (c_1 x^{d_e}, \dots, c_5 x^{d_e}, c y^{d_e}) \in H^0(\mathcal{O}_{\mathbb{P}^1}(d_e))^{\oplus 6}, \quad (c_1, \dots, c_5) \neq 0, \quad c \neq 0,$$

where $[x, y]$ are homogeneous coordinates of \mathbb{P}^1 . An arrow from $(c_i x^{d_e}, c y^{d_e})$ to $(c'_i x^{d_e}, c' y^{d_e})$ consists of an isomorphism $\sigma : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ fixing $[1, 0]$ and $[0, 1]$, and an isomorphism $\mathcal{O}_{\mathbb{P}^1}(d_e) \cong \sigma^* \mathcal{O}_{\mathbb{P}^1}(d_e)$ that identifies $c_i x^{d_e}$ with $c'_i x^{d_e}$, and identifies $c y^{d_e}$ with $c' y^{d_e}$.

Lemma 2.30. *For $e \in E_0(\Gamma)$, we have the following morphism and isomorphism defined by Lemma 2.11:*

$$\Phi_e : \mathcal{W}_\Gamma \longrightarrow \mathcal{W}_e \cong \sqrt[d_e]{\mathcal{O}_{\mathbb{P}^4}(1)/\mathbb{P}^4}.$$

Proof. Firstly, we prove the isomorphism. We form $[(\mathbb{C}^5 \setminus 0)/\mathbb{G}_m]$, where $\alpha \in \mathbb{G}_m$ acts on $\mathbb{C}^5 \setminus 0$ by $\alpha \cdot (c_1, \dots, c_5) = (\alpha^{d_e} c_1, \dots, \alpha^{d_e} c_5)$. We first show that the map sending (2.13) to $[c^{-1} c_1, \dots, c^{-1} c_5] \in [(\mathbb{C}^5 \setminus 0)/\mathbb{G}_m]$ is well-defined. Firstly, it is independent of the choice of the isomorphism $\mathcal{L}|_{\mathbb{C}_e} \cong \mathcal{O}_{\mathbb{P}^1}(d_e)$. Indeed, different isomorphisms will scale both c and (c_1, \dots, c_5) by a non-zero constant leaving $(c^{-1} c_1, \dots, c^{-1} c_5)$ invariant. On the other hand, given an automorphism of \mathbb{P}^1 of the form $\alpha \cdot [x, y] = [\alpha^{-1} x, y]$, then

$$\alpha \cdot (c_1 x^{d_e}, \dots, c_5 x^{d_e}, c y^{d_e}) = (\alpha^{-d_e} c_1 x^{d_e}, \dots, \alpha^{-d_e} c_5 x^{d_e}, c y^{d_e}).$$

This proves that the map $\mathcal{W}_e \rightarrow [(\mathbb{C}^5 \setminus 0)/\mathbb{G}_m]$ is well-defined and is an isomorphism. Using

$$[(\mathbb{C}^5 \setminus 0)/\mathbb{G}_m] \cong \sqrt[d_e]{\mathcal{O}_{\mathbb{P}^4}(1)/\mathbb{P}^4},$$

we prove the isomorphism part of the Lemma.

We now define the morphism Φ_e . Let (ξ_S, ϵ_S) be an S -family in \mathcal{W}_Γ , and let $s \in S$ be a closed point. Denote $\xi_S = (\mathcal{C}_S, \Sigma^{\mathcal{C}_S}, \mathcal{L}_S, \dots)$. Following the notation developed before and in the proof of Lemma 2.11, let $q = \mathcal{C}_{s,e} \cap (\mathcal{C}_s)_0$ and $q' = \mathcal{C}_{s,e} \cap (\mathcal{C}_s)_1$. Consider the case where q and q' are the two nodes of \mathcal{C}_s in $\mathcal{C}_{s,e} (= (\mathcal{C}_s)_e)$. The proof for other cases are similar. As both q and q' are \mathbb{G}_m -unbalanced, and because (ξ_S, ϵ_S) is a Γ -framed MSP field, q and q'

extend to S -families of nodes $\mathcal{Q}_S, \mathcal{Q}'_S \subset \mathcal{C}_S$ containing q and q' respectively, which decompose \mathcal{C}_S along \mathcal{Q}_S and \mathcal{Q}'_S into two families of curves, one of which is a family of rational curves $\mathcal{C}_{S,e}$ indexed by $e \in E_0(\Gamma)$ with two sections \mathcal{Q}_S and $\mathcal{Q}'_S \subset \mathcal{C}_{S,e}$. By shrinking S with $s \in S$ if necessary, we can find an S -isomorphism $\phi : \mathbb{P}^1 \times S \cong \mathcal{C}_{S,e}$ so that $\phi^{-1}(\mathcal{Q}_S) = [1, 0] \times S$ and $\phi^{-1}(\mathcal{Q}'_S) = [0, 1] \times S$. We then fix an isomorphism $\phi^* \mathcal{L}_S|_{\mathcal{C}_{S,e}} \cong \pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(d_e)$, and, by Lemma 2.11, express

$$(2.14) \quad \phi^* \varphi_{i,S}|_{\mathcal{C}_{S,e}} = c_i x^{d_e}, \quad \phi^* \nu_{1,S}|_{\mathcal{C}_{S,e}} = c y^{d_e},$$

where c_i, c are regular functions on S so that (c_1, \dots, c_5) and c are nowhere vanishing. The S -family (2.14) defines a morphism $S \rightarrow \mathcal{W}_e$. Clearly, different choices of isomorphisms ϕ and isomorphism $\phi^* \mathcal{L}_S|_{\mathcal{C}_{S,e}} \cong \pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(d_e)$ produce another $S \rightarrow \mathcal{W}_e$ that differs from the previous $S \rightarrow \mathcal{W}_e$ by an arrow between them. Thus we obtain a morphism $\Phi_e(\xi_S, \epsilon_S) : S \rightarrow \mathcal{W}_e$, which patches to form a morphism $\Phi_e : \mathcal{W}_\Gamma \rightarrow \mathcal{W}_e$, as desired. \square

To study the structure of the morphism Φ , we investigate the associated moduli of $e \in E(\Gamma)$. Let $e \in E_\infty(\Gamma)$. Recall the statement before Lemma 2.12, let $v \in V_\infty(\Gamma)$ and $v' \in V_1(\Gamma)$ be so that (e, v) and $(e, v') \in F(\Gamma)$. Set $\delta = -1$ when $v \in V_\infty^{0,1}(\Gamma)$, and $\delta = 0$ otherwise, set $\delta' = -1$ when $v' \in V_1^{0,1}(\Gamma)$ and $\delta' = 0$ otherwise. Note that when $d_e = d_{0e} - d_{\infty e} \notin \mathbb{Z}$, $\delta = 0$.

We introduce \mathcal{W}_e . In case $d_{\infty e} \notin \mathbb{Z}$, let $(\mathcal{X}, q, q') = (\mathbb{P}(5, 1), [1, 0], [0, 1])$, etc., be as in and before Lemma 2.12. Let $\mathcal{N} = \mathcal{O}_{\mathcal{X}}(-d_e)$, and let $\omega = \mathcal{O}_{\mathcal{X}}(\delta' q') \cong \mathcal{O}_{\mathcal{X}}(\delta')$. (Note $\delta = 0$ in this case.) We define \mathcal{W}_e to be the stack parameterizing families of

$$(2.15) \quad (c x^{-5d_e + \delta + \delta'}, c' y^{\lfloor -d_e \rfloor}) \in H^0(\mathcal{N}^{\otimes 5} \otimes \omega \oplus \mathcal{N}), \quad c, c' \in \mathbb{C}^\times,$$

where an arrow from $(c_1 x^{-5d_e + \delta'}, c'_1 y^{\lfloor -d_e \rfloor})$ to $(c_2 x^{-5d_e + \delta'}, c'_2 y^{\lfloor -d_e \rfloor})$ consists of an isomorphism $\sigma_1 : \mathcal{X} \rightarrow \mathcal{X}$ fixing q and q' , and an isomorphism $\sigma_2 : \mathcal{N} \cong \sigma_1^* \mathcal{N}$ which together with the tautological isomorphism² $\sigma_1^* \omega \cong \omega$ identifies $(c_1 x^{-5d_e + \delta'}, c'_1 y^{\lfloor -d_e \rfloor})$ with $(c_2 x^{-5d_e + \delta'}, c'_2 y^{\lfloor -d_e \rfloor})$.

When $d_e \in \mathbb{Z}$, let $(X, q, q') = (\mathbb{P}^1, [1, 0], [0, 1])$ under some homogeneous coordinates $[x, y]$. Let $\mathcal{N} = \mathcal{O}_X(-d_e)$, and let $\omega = \mathcal{O}_X(\delta q + \delta' q')$. We define \mathcal{W}_e to be the moduli stack parameterizing families of (2.15), where arrows between objects are defined similarly.

Lemma 2.31. *Let $e \in E_\infty(\Gamma_\xi)$. Then \mathcal{W}_e is a $\mu_{-5d_e + \delta}$ -gerbe over a point.*

Proof. First, by scaling the line bundle \mathcal{L} and applying automorphisms to X preserving (q, q') , we can make $(c, c') = (1, 1)$. Thus \mathcal{W}_e as a set consists of a single point.

Let G_e be the automorphism group of the single point in \mathcal{W}_e . Thus \mathcal{W}_e is a G_e -gerbe over a single point. We now prove that $G_e \cong \mu_{-5d_e + \delta}$. We only

²Here the tautological map $\sigma_1^* \omega \cong \omega$ is consistent with the tautological inclusion $\omega \subset \mathcal{O}_{\mathcal{X}}$ and the isomorphism $\sigma_1^* \mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$ fixing the global section 1.

prove the case where $d_e \notin \mathbb{Z}$ while the other case is similar. Let $\sigma \in G_e$, let $\sigma_1 : \mathcal{X} \rightarrow \mathcal{X}$ be the automorphism part of σ , and let $\sigma_2 : \sigma_1^* \mathcal{N} \rightarrow \mathcal{N}$ be the linearization part of σ . Since $c'y^{\lfloor -d_e \rfloor}|_{q'} \neq 0$ and σ_1 fixes q' , $\sigma \in G_e$ implies that $\sigma_2|_{q'} : \mathcal{N}|_{q'} = \sigma_1^* \mathcal{N}|_{q'} \rightarrow \mathcal{N}|_{q'}$ is the identity map. In particular, the map $G_e \rightarrow \text{Aut}(\mathcal{X})$ is injective. Furthermore, let $\sigma'_2 : \sigma_1^* \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}}$ be the G_e -linearization leaving $1 \in H^0(\mathcal{O}_{\mathcal{X}})$ invariant, then the linearization σ_2 is that induced by a G_e -isomorphism $\mathcal{N} \cong \mathcal{O}_{\mathcal{X}}(-5d_e q)$, the linearization σ'_2 on $\mathcal{O}_{\mathcal{X}}$, and the tautological inclusion

$$(2.16) \quad \mathcal{O}_{\mathcal{X}} \subset \mathcal{O}_{\mathcal{X}}(-5d_e q) \cong \mathcal{N}.$$

Then since the meromorphic section $x^{\delta'}$ of ω is the meromorphic section 1 in the footnote below (2.15), $x^{\delta'}$ is \mathbb{C}^* -equivariant. Thus $x^{-5d_e + \delta'} \in H^0(\mathcal{N}^{\otimes 5} \otimes \omega)$ is G_e -invariant if and only if $x^{-5d_e} \in H^0(\mathcal{N}^{\otimes 5})$ is G_e -invariant. Using the G_e -equivariant homomorphisms in (2.16), we see that $G_e = \mu_{-5d_e} \leq \mathbb{C}^*$, which is $\mu_{-5d_e + \delta}$ as $\delta = 0$ in this case. \square

Corollary 2.32. *The families given in Lemma 2.12 induce a morphism*

$$\Phi_e : \mathcal{W}_{\Gamma} \longrightarrow \mathcal{W}_e \cong B\mu_{-5d_e + \delta}.$$

Remark 2.33. *The proof of Lemma 2.31 shows that for $e \in E_{\infty}(\Gamma_{\xi})$ with $d_e \notin \mathbb{Z}$, a generator of the automorphisms of $\xi|_{\mathcal{C}_e}$ takes the following form near its stacky point q : Let $[\tilde{x}, \tilde{y}]$ be a homogeneous coordinate of \mathcal{C}_e with $q = [1, 0]$ and $q' = [0, 1]$, let $\mathcal{U}_e = \mathcal{C}_e - q'$, and let $b = \lfloor -d_e \rfloor$ and $m = 5b + 5d_e$. Then $\mathcal{L}|_{\mathcal{U}_e} \cong \tilde{y}^{-m} \mathbb{C}[y]$ (compare with that before (2.18)), and a generator of $\text{Aut}(\xi|_{\mathcal{C}_e})$ acts on $\mathcal{L}|_{\mathcal{U}_e}$ via*

$$(2.17) \quad [a, c] \mapsto [\zeta_l \cdot a, c] \in C_e \quad \text{and} \quad 1 \mapsto \zeta_l^b \cdot 1.$$

Here $\ell = -5d_e$.

The remark can be seen as follows: The action on \mathcal{C}_e is the same as its action on the coarse moduli C_e , and the action is as stated. Using $\mathcal{L}|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathcal{C}_e}((m - 5b)q)$, we see that $\mathcal{L}|_{\mathcal{U}_e} \cong \tilde{y}^{-m} \mathbb{C}[y]$ is equivalent to $\mathcal{O}_{\mathcal{C}_e}(-5bq)|_{\mathcal{U}_e} \cong \mathbb{C}[y]$. As $TC_e|_{[1,0]} \cong \mathbf{L}_{-1}^3$, $TC_e|_{[0,1]} \cong \mathbf{L}_1$, and $\mathcal{L}|_{q'} \cong \mathbf{L}_0$ from the proof of Lemma 2.31, we get $\mathcal{O}_{\mathcal{C}_e}(-5bq)|_q \cong \mathbf{L}_b$. As $1 \in \mathbb{C}[y]$ in the stated isomorphism spans $\mathcal{O}_{\mathcal{C}_e}(-5bq)|_q$, we get $1 \mapsto \zeta_l^b \cdot 1$.

Proposition 2.34. *The morphism $\Phi : \mathcal{W}_{\Gamma} \rightarrow \mathcal{W}_{\Gamma^{\text{de}}}$ in Lemma 2.27 is an isomorphism.*

Before proving the proposition, we state simple facts on lifting automorphisms of a twisted curve to their invertible sheaves. We recall notations used in [CLLL, Subsect. 2.1] (drawn from [ACV, AF, AGV, Cad].)

First a model of an index r balanced node is

$$\mathcal{V}_r := [\text{Spec}(\mathbb{C}[u, v]/(uv)) / \mu_r], \quad \zeta \cdot (u, v) = (\zeta u, \zeta^{-1} v).$$

³By this we mean it is a weight -1 representation of μ_l .

An index r marking of a twisted curve looks like the model

$$\mathcal{U}_r := [\mathrm{Spec} \mathbb{C}[u]/\mu_r], \quad \zeta \cdot u = \zeta u.$$

Invertible sheaves (representable) on the model \mathcal{V}_r is a μ_r -module \mathcal{M}_m , $0 < m < r$:

$$\mathcal{M}_m := u^{-(r-m)}\mathbb{C}[u] \oplus_{[0]} v^{-m}\mathbb{C}[v] := \ker\{u^{-(r-m)}\mathbb{C}[u] \oplus v^{-m}\mathbb{C}[v] \rightarrow \mathbb{C}_m\},$$

where the arrow is a homomorphism of μ_r -modules, μ_r leaves $1 \in \mathbb{C}[u]$ and $1 \in \mathbb{C}[v]$ fixed and acts on $1 \in \mathbb{C}_m \cong \mathbb{C}$ via $\zeta \cdot 1 = \zeta^m 1$, and both $u^{-(r-m)}\mathbb{C}[u] \rightarrow \mathbb{C}_m$ and $v^{-m}\mathbb{C}[v] \rightarrow \mathbb{C}_m$ are surjective. Under this convention,

$$(2.18) \quad \pi_{r*}\mathcal{M}_m = \mathbb{C}[x] \oplus \mathbb{C}[y],$$

where π_r is the projections to its coarse moduli space

$$(2.19) \quad \begin{aligned} \pi_r : \mathcal{V}_r &\rightarrow V_r := \mathrm{Spec}(\mathbb{C}[x, y]/(xy), \\ \text{and } \pi_r : \mathcal{U}_r &\rightarrow U_r := \mathrm{Spec} \mathbb{C}[x], \quad x = u^r, y = v^r. \end{aligned}$$

In the following, we will repeatedly use the fact that if we let $1_u \in \mathbb{C}[u]$ be the element $1 \in \mathbb{C} \subset \mathbb{C}[u]$, and let 1_v , 1_x and 1_y be similarly defined elements, then π_{r*} in (2.18) sends 1_u and 1_v to 1_x and 1_y , respectively.

Note that the relative automorphism groups $\mathrm{Aut}_{U_r}(\mathcal{U}_r) \cong \{1\}$, $\mathrm{Aut}_{V_r}(\mathcal{V}_r) \cong \mu_r$, where the latter is generated by $\zeta_r \cdot (u, v) = (u, \zeta_r^{-1}v)$. These groups are useful in our study of MSP fields due to the following lifting property. We assume r is an odd prime. We will follow the notations before and after (2.19).

Lemma 2.35. *Let $m \in [1, r-1]$. There is a unique $\mathrm{Aut}_{V_r}(\mathcal{V}_r)$ -linearization of the module \mathcal{M}_m so that it leaves $1_u \in \mathcal{M}_m$ fixed.*

Proof. Let $\alpha \in \mathrm{Aut}_{V_r}(\mathcal{V}_r)$ so that $\alpha \cdot (u, v) = (u, \zeta_r^{-1}v)$. To make \mathcal{M}_m an $\mathrm{Aut}_{V_r}(\mathcal{V}_r)$ -module that leaves 1_u fixed, we only need to find $\alpha \cdot 1_v = \zeta_r^a 1_v$ so that α acts on $u^{-(r-m)}1_u$ and $v^{-m}1_v$ by identical factors. Since $\alpha \cdot (u^{-(r-m)}1_u) = u^{-(r-m)}1_u$ and $\alpha \cdot (v^{-m}1_v) = \zeta_r^m \zeta_r^a 1_v$, they are identical if and only if $1 = \zeta_r^m \zeta_r^a$. As $m \in [1, r-1]$, it is uniquely solvable (mod r) by $a = r - m$. \square

A simple corollary of this is that the only $\alpha \in \mathrm{Aut}_{V_r}(\mathcal{V}_r)$ that lifts to an automorphism \mathcal{M}_m fixing $(1_u, 1_v)$ is the trivial element $\alpha = 1 \in \mathrm{Aut}_{V_r}(\mathcal{V}_r)$.

Lemma 2.36. *Let $b \geq 0$ and $m \in [1, r-1]$ be integers so that $(m, r) = 1$, and let $l = rb + m$. Given a μ_l -action on V_r via $\zeta_l \cdot (x, y) = (x, \zeta_l^{-1}y)$ and a μ_l -linearization on $\pi_{r*}\mathcal{M}_m = \mathbb{C}[x] \oplus \mathbb{C}[y]$ via $\zeta_l \cdot (1_x, 1_y) = (1_x, \zeta_l^b 1_y)$, there is a unique lifting of this μ_l -action to \mathcal{V}_r together with a μ_l -linearization of \mathcal{M}_m covering that of $\pi_{r*}\mathcal{M}_m$.*

Proof. A general extension of $\zeta_l \cdot (x, y) = (x, \zeta_l^{-1}y)$ is $\zeta_l \cdot (u, v) = (\zeta_r^c u, \zeta_r^{-1} \zeta_r^{c'} v)$ for some $c, c' \in \mathbb{Z}$. Since the μ_l -action on \mathcal{V}_r via $\zeta_r \cdot (u, v) = (\zeta_r^{c'} u, \zeta_r^{-c'} v)$ is the trivial action, by replacing c with $c + c'$, we can assume $c' = 0$. Then

$$\zeta_l \cdot (u^{-(r-m)}1_u, v^{-m}1_v) = (\zeta_r^{-c(r-m)}u^{-(r-m)}1_u, \zeta_r^m \zeta_l^b v^{-m}1_v).$$

It lifts to an action on \mathcal{M}_m if and only if $\zeta_r^{-c(r-m)} = \zeta_{rl}^m \zeta_l^b = \zeta_{rl}^{rb+m}$. Using $l = rb + m$, the latter identity is equivalent to $\zeta_r^{cm} = \zeta_r$. Because $(m, r) = 1$, it is uniquely solvable for an integer $c \in [1, r-1]$. This proves the lemma. \square

Proof of Proposition 2.34. We follow the notation in the proof of Proposition 2.27. Let $n \in N(\Gamma)$ and let Γ' be the decoupling of Γ along n . We look at the tautological morphism $\phi : \mathcal{W}_\Gamma \rightarrow \mathcal{W}_{\Gamma'}$ in (2.12). We will show that ϕ is an isomorphism.

We first consider the case where $v \in V_\infty^S(\Gamma)$. After a moment of thought, we see that we only need to show a simplified case where $E(\Gamma)$ contains one edge $e \in E_\infty(\Gamma)$, and $V(\Gamma)$ contains a stable vertex $v \in V_\infty^S(\Gamma)$, and one unstable vertex $v' \in V_1(\Gamma)$. In this case, the morphism ϕ is

$$\phi : \mathcal{W}_\Gamma \longrightarrow \mathcal{W}_v \times \mathcal{W}_e.$$

When $d_e \in \mathbb{Z}$, it is obvious. From now on, we assume $d_e \notin \mathbb{Z}$.

We prove that ϕ is an isomorphism of stacks. First, it is easy to see that ϕ is one-one, onto and étale. Thus to prove the Proposition we only need to show that for any $(\xi, \epsilon) \in \mathcal{W}_\Gamma$, $\text{Aut}((\xi, \epsilon)) \cong \text{Aut}(\phi(\xi, \epsilon))$.

For $a \in \{v, e\}$, let $\xi_a = \xi|_{\mathcal{C}_a}$. Any $\iota \in \text{Aut}((\xi, \epsilon))$ induces an $\iota_a \in \text{Aut}(\xi_a)$ for every $a \in \{a, v\}$. We claim that if all $\iota_a = \text{id}$, then $\iota = \text{id}$. Indeed, let $\alpha_0 \in \text{Aut}(\mathcal{C})$ be the image of ι in $\text{Aut}(\mathcal{C})$, and let α_1 and α_2 be the associated isomorphisms $\alpha_{0*}\mathcal{L} \rightarrow \mathcal{L}$ and $\alpha_{0*}\mathcal{N} \rightarrow \mathcal{N}$. Assuming $\iota_a = \text{id}$ for $a = e$ and v , then $\alpha_0|_{\mathcal{C}_a} = \alpha_1|_{\mathcal{C}_a} = \alpha_2|_{\mathcal{C}_a} = \text{id}$. Therefore, α_0 induces the identity in $\text{Aut}(\mathcal{C})$. Suppose $\alpha_0 \neq \text{id}$, then $\alpha_0 \in \text{Aut}_C(\mathcal{C})$. Because ξ is representable (cf. Definition 2.1), by Lemma 2.35, α_0 can not be lifted to automorphisms of both \mathcal{L} and \mathcal{N} so that their restrictions over the scheme part of \mathcal{C} are identities. Therefore, $\alpha_0 = \text{id}$. Then, applying the uniqueness part of Lemma 2.35, we see that $\alpha_1 = \alpha_2 = \text{id}$. This proves that $\iota_e = \iota_v = \text{id}$ imply $\iota = \text{id}$.

We now prove the converse: If $\{\kappa_e, \kappa_v\} \in \text{Aut}(\phi((\xi, \epsilon)))$, then we can find an $\iota \in \text{Aut}((\xi, \epsilon))$ so that $\iota_e = \kappa_e$ and $\iota_v = \kappa_v$. Clearly, it suffices to show that this is true when one of $\{\kappa_e, \kappa_v\}$ is trivial.

We first look at the case when $\kappa_e \neq \text{id}$. Let $q = \mathcal{C}_e \cap \mathcal{C}_v$ and $q' = \mathcal{C}_{v'}$, viewed as a point in \mathcal{C}_e . Let $\alpha_{e0} \in \text{Aut}(\mathcal{C}_e)$ be its induced automorphism, which leaves q and q' fixed. Let

$$(2.20) \quad \alpha_{e1} : \alpha_{e0*}\mathcal{L}|_{\mathcal{C}_e} \rightarrow \mathcal{L}|_{\mathcal{C}_e} \quad \text{and} \quad \alpha_{e2} : \alpha_{e0*}\mathcal{N}|_{\mathcal{C}_e} \rightarrow \mathcal{N}|_{\mathcal{C}_e}$$

be the associated isomorphisms. Since $d_{\infty e} \notin \mathbb{Z}$, near q , we can apply Remark 2.33 and Lemma 2.36 to conclude that we can extend α_{e0} to an automorphism α_{ve0} of $\mathcal{C}_{ev} := \mathcal{C}_e \cup \mathcal{C}_v$, and extend α_{e1} and α_{e2} to α_{ev1} and α_{ev2} , like (2.20) with v replaced by ve . so that $\iota_{ev} = \{\alpha_{evi}\}$ becomes an automorphism of $(\xi, \epsilon)|_{\mathcal{C}_{ev}}$ with $\iota_{ev}|_{\mathcal{C}_v} = \text{id}$. Because of this, and because $\alpha_{e1}|_{q'} = \alpha_{e2}|_{q'} = \text{id}$, we can further extend ι_{ev} to an automorphism ι of (ξ, ϵ) .

At last, we consider the case where $\kappa_v \neq \text{id}$ but $\kappa_e = \text{id}$. We claim that we can extend κ_v to a $\kappa_{ve} \in \text{Aut}((\xi, \epsilon)|_{\mathcal{C}_{ve}})$ so that $\kappa_{ve}|_{\mathcal{C}_e} = \text{id}$. Let $(\alpha_{v0}, \alpha_{v1}, \alpha_{v2})$

be as before; note that since $d_e \notin \mathbb{Z}$, $q = \mathcal{C}_v \cap \mathcal{C}_e$ is a stack point. Without loss of generality, we can assume that there is an α_{v0} -invariant étale chart $\mathcal{U}_v = [U'_v/\mu_5] \rightarrow \mathcal{C}_v$ so that $\alpha_{v0}|_{\mathcal{U}_v}$ lifts to $\alpha'_{v0} : U'_v \rightarrow U'_v$, and that there is an $\tilde{x} \in \Gamma(\mathcal{O}_{U'_v})$ so that $\zeta_5 \cdot \tilde{x} = \zeta_5 \tilde{x}$ and $q = [(\tilde{x} = 0)/\mu_5]$. Because $\rho|_{\mathcal{C}_v}$ is nowhere vanishing, we can assume that $\mathcal{L}|_{U'_v} \cong \tilde{x}^{-(5-m)} \mathbb{C}[\tilde{x}]$, for an $m \in [1, 4]$, so that $\rho|_{U'_v} = (\tilde{x}^{-(5-m)} 1)^5$. Since ρ is invariant under κ_v , $\bar{\alpha}_{v1} := \alpha_{v1} \circ \alpha_{v0}^*$ fixes $(\tilde{x}^{-(5-m)} 1)^5$. Thus there is a $c \in [0, 4]$ so that $\bar{\alpha}_{v1}$ sends $\tilde{x}^{-(5-m)} 1$ to $\zeta_5^c \tilde{x}^{-(5-m)} 1$.

Now let α_{ve0} be an extension of α_{v0} to $\mathcal{C}_v \cup \mathcal{C}_e$ so that $\alpha_{ve0}|_{\mathcal{C}_e} = \text{id}$. By an argument similar to the proof of Lemma 2.35, we can extend α_{v1} to an $\alpha_{ve1} : \alpha_{ve0*} \mathcal{L}|_{\mathcal{C}_{ve}} \rightarrow \mathcal{L}|_{\mathcal{C}_{ve}}$. Because $\bar{\alpha}_{v1}$ sends $\tilde{x}^{-(5-m)} 1$ to $\zeta_5^c \tilde{x}^{-(5-m)} 1$, by the proof of the same Lemma, we see that there is a $c' \in [0, 4]$ so that $\alpha_{ve1}|_{\mathcal{C}_e} = \zeta_5^{c'} \text{id}$.

When $c' = 0$, we are done. Otherwise, we apply Lemma 2.35 to pick a $\tau \in \text{Aut}_{C_{ve}}(\mathcal{C}_{ve})$ so that it lifts to an action τ (the same letter for simplicity) on $\mathcal{L}|_{\mathcal{C}_{ve}}$ so that $\tau|_{\mathcal{C}_v} = \text{id}$ and $\tau|_{\mathcal{C}_e} = \zeta_5^{-c'} \text{id}$. Let $\beta_{ve0} = \tau \circ \alpha_{ve0}$. Then β_{ve0} lifts to a $\beta_{ve1} : \beta_{ve0*} \mathcal{L}|_{\mathcal{C}_{ve}} \rightarrow \mathcal{L}|_{\mathcal{C}_{ve}}$ so that $\beta_{ve1}|_{\mathcal{C}_v} = \alpha_{v1}$ and $\beta_{ve1}|_{\mathcal{C}_e} = \text{id}$. As to \mathcal{N} , α_{v2} extends to β_{ve2} that is α_{v2} and id when restricted to \mathcal{C}_v and \mathcal{C}_e , respectively. Because the field φ vanishes along q , we see that $(\varphi, \rho, \nu_1, \nu_2)|_{\mathcal{C}_{ve}}$ are invariant under $\kappa_{ve} := (\beta_{ve0}, \beta_{ve1}, \beta_{ve2})$.

Therefore, applying the previous arguments, we conclude that we can extend κ_v to an $\iota \in \text{Aut}((\xi, \epsilon))$ so that $\iota_e = \text{id}$. This proves that ϕ is an isomorphism when $v \in V_\infty^S(\Gamma)$.

The case when $v \in V_1(\Gamma) \cup V_\infty^{0,2}(\Gamma)$ is similar but simpler. We will omitted the details. This proves the proposition. \square

For a regular Γ (cf. Def. 2.28), the decomposition $\mathcal{W}_{\Gamma^{\text{de}}}$ takes a particular simple form. For an unstable $v \in V_0^U(\Gamma)$, define $\mathcal{W}_v = \mathbb{P}^4$. In general, for $v \in V_0(\Gamma)$, define

$$\mathcal{W}_{[v]} := \mathcal{W}_v \times_{(\mathbb{P}^4)^{E_v}} \prod_{e \in E_v} \mathcal{W}_e,$$

where $\mathcal{W}_v \rightarrow (\mathbb{P}^4)^{|E_v|}$ is the product of the evaluation maps $\text{ev}_{(e,v)} : \mathcal{W}_v \rightarrow \mathbb{P}^4$. For $e \in E_v$, $\mathcal{W}_e \rightarrow \mathbb{P}^4$ is the coarse moduli morphism given by Lemma 2.30.

By the previous arguments, we have

Corollary 2.37. *Let Γ be a regular graph. Then we have a canonical isomorphism*

$$\mathcal{W}_{\Gamma^{\text{de}}} \cong \prod_{v \in V_0(\Gamma)} \mathcal{W}_{[v]} \times \prod_{v \in V_1^S(\Gamma)} \mathcal{W}_v \times \prod_{e \in E_\infty(\Gamma)} \mathcal{W}_e \times \prod_{v \in V_\infty^S(\Gamma)} \mathcal{W}_v.$$

3. VIRTUAL LOCALIZATION, PART 1

Recall that for a DM T -stack W with a T -equivariant perfect obstruction theory $\phi_W^\vee : \mathbb{T}_W \rightarrow E_W$ and a T -invariant cosection $\sigma : \mathcal{O}_W \rightarrow \mathcal{O}_W$, by [GP],

the T -invariant part of the perfect obstruction theory induces a perfect obstruction theory of the fixed locus W^T with the obstruction sheaf $(\mathcal{O}b_W|_{W^T})^T$ and the cosection (the invariant part of σ)

$$\sigma^T : \mathcal{O}b_{W^T} := (\mathcal{O}b_W|_{W^T})^T \longrightarrow \mathcal{O}_{W^T}.$$

Let $\coprod_{a \in \Lambda} W_a = W^T$ be a decomposition into disjoint open and closed substacks, let $\sigma_a = \sigma^T|_{W_a}$. Let $D(\sigma)$ be the degeneracy (non-surjective) part of σ , let $\iota_a : D(\sigma_a) \rightarrow D(\sigma)$ be the inclusion, and let

$$[W]_{\text{loc}}^{\text{vir}} \in A_*^T D(\sigma) \quad (\text{resp. } [W_a]_{\text{loc}}^{\text{vir}} \in A_*^T D(\sigma_a))$$

be the cosection localized equivariant virtual cycle of (W, σ) (resp. (W_a, σ_a)). The following virtual localization of cosection localized virtual cycles (analogous to [GP]) is addressed in [CKL].

Theorem 3.1 (Virtual localization formula). *Let the notations be as stated. Suppose each moving part $(E_W|_{W_a})^{\text{mv}}$ is quasi-isomorphic to a T -equivariant two-term complex of coherent locally free sheaves $[\mathcal{F}_{a,0} \rightarrow \mathcal{F}_{a,1}]$. Then after inverting (the generator) $\mathfrak{t} \in H^2(BT)$ and letting $e(N_a^{\text{vir}}) = e(\mathcal{F}_{a,0})/e(\mathcal{F}_{a,1})$, we have*

$$[W]_{\text{loc}}^{\text{vir}} = \sum_{a \in \Lambda} \iota_{a*} \left(\frac{[W_a]_{\text{loc}}^{\text{vir}}}{e(N_a^{\text{vir}})} \right) \in A_*^T D(\sigma) [\mathfrak{t}^{-1}].$$

3.1. Applying virtual localization formula. To apply the localization formula, we first lift a T -equivariant relative obstruction theory of $\mathcal{W}(= \mathcal{W}_{g,\gamma,\mathbf{d}})$ to a T -equivariant perfect obstruction theory of \mathcal{W} .

We recall the relevant notations. Recall that \mathcal{D} is the stack of $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N})$. By forgetting (φ, ρ, ν) , we get the forgetful morphism $q : \mathcal{W} \rightarrow \mathcal{D}$. Let

$$(\mathcal{C}_{\mathcal{W}}, \Sigma^{\mathcal{C}_{\mathcal{W}}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu) \quad \text{and} \quad \pi : \mathcal{C}_{\mathcal{W}} \longrightarrow \mathcal{W}$$

be the universal family of \mathcal{W} . Following [CLLL], we have the relative perfect obstruction theory

$$(3.1) \quad \phi_{\mathcal{W}/\mathcal{D}}^{\vee} : \mathbb{T}_{\mathcal{W}/\mathcal{D}} \longrightarrow \mathbb{E}_{\mathcal{W}/\mathcal{D}} := R\pi_* \mathcal{V},$$

where we set $\mathcal{P} = \mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}_{\mathcal{W}}/\mathcal{W}}^{\log}$, set

$$(3.2) \quad \mathcal{L}^{\log} = \mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathcal{C}_{\mathcal{W}}}), \quad \mathcal{P} = \mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}_{\mathcal{W}}/\mathcal{W}}^{\log}, \quad \text{and} \quad \mathcal{P}^{\log} = \mathcal{P}(-\Sigma_{(1,\rho)}^{\mathcal{C}_{\mathcal{W}}}),$$

and set

$$(3.3) \quad \mathcal{V} = (\mathcal{L}^{\log})^{\oplus 5} \oplus \mathcal{P}^{\log} \oplus \mathcal{L} \otimes \mathcal{N} \otimes \mathbf{L}_1 \oplus \mathcal{N}.$$

It comes with a cosection $\sigma_{\mathcal{W}/\mathcal{D}} : \mathcal{O}b_{\mathcal{W}/\mathcal{D}} \rightarrow \mathcal{O}_{\mathcal{W}}$. Following the constructions in [CL1], both the obstruction theory and the cosection are T -equivariant, and liftable to a cosection of $\mathcal{O}b_{\mathcal{W}}$, thus provide a T -equivariant cycle $[W]_{\text{loc}}^{\text{vir}}$.

To apply the virtual localization formula, we first construct a T -equivariant (absolute) obstruction theory $\phi_{\mathcal{W}}^{\vee}$. Let θ be the arrow in the distinguished triangle (d.t. for short) in the lower line in (3.4) induced by $q : \mathcal{W} \rightarrow \mathcal{D}$.

$$(3.4) \quad \begin{array}{ccccccc} \longrightarrow & q^* \mathbb{T}_{\mathcal{D}}[-1] & \xrightarrow{\phi_{\mathcal{W}/\mathcal{D}}^{\vee} \circ \theta} & \mathbb{E}_{\mathcal{W}/\mathcal{D}} & \longrightarrow & \mathbb{E}_{\mathcal{W}} & \xrightarrow{+1} \longrightarrow \\ & \parallel & & \uparrow \phi_{\mathcal{W}/\mathcal{D}}^{\vee} & & \uparrow \phi_{\mathcal{W}}^{\vee} & \\ \longrightarrow & q^* \mathbb{T}_{\mathcal{D}}[-1] & \xrightarrow{\theta} & \mathbb{T}_{\mathcal{W}/\mathcal{D}} & \longrightarrow & \mathbb{T}_{\mathcal{W}} & \xrightarrow{+1} \longrightarrow \end{array}$$

Here $\mathbb{E}_{\mathcal{W}}$ is the mapping cone of $\phi_{\mathcal{W}/\mathcal{D}}^{\vee} \circ \theta$, and $\phi_{\mathcal{W}}^{\vee}$ (shown above) is an arrow in $D^+(\mathcal{O}_{\mathcal{W}})$ that makes above an arrow of d.t.'s. A standard argument⁴ shows that $\phi_{\mathcal{W}}^{\vee}$ is a perfect obstruction theory.

To proceed, we argue that we can make objects and arrows in (3.4) lie in $D_{\text{qcoh}}^+(\mathcal{O}_{[\mathcal{W}/T]})$, where $D_{\text{qcoh}}^+(\mathcal{O}_{[\mathcal{W}/T]})$ is the subcategory of $D(\mathcal{O}_{[\mathcal{W}/T]})$ consisting of complexes of $\mathcal{O}_{[\mathcal{W}/T]}$ -modules with quasi-coherent cohomologies [LM, Definition 3.1].

Applying [LM, Prop 13.2.6-iii] to $\pi_{\mathcal{W}} : \mathcal{C}_{\mathcal{W}} \rightarrow \mathcal{W}$, the construction of $\phi_{\mathcal{W}/\mathcal{D}}^{\vee}$ shows that $\phi_{\mathcal{W}/\mathcal{D}}^{\vee}$ can be represented as an arrow in $D_{\text{qcoh}}^+(\mathcal{O}_{[\mathcal{W}/T]})$ in the *lisse-étale* site. Here we used the fact that $\mathcal{C}_{\mathcal{W}}$ admits a T -equivariant étale open affine atlas, which implies that the category of quasi-coherent sheaves over $[\mathcal{C}_{\mathcal{W}}/T]_{\text{lisse-étale}}$ is equivalent to the category of quasi-coherent T -equivariant sheaves over $(\mathcal{C}_{\mathcal{W}})_{\text{lisse-étale}}$. The same holds for \mathcal{W} .

By Illusie's construction of cotangent complexes, the θ in (3.4) is an arrow in $D_{\text{qcoh}}^+(\mathcal{O}_{[\mathcal{W}/T]})$. Thus $\phi_{\mathcal{W}/\mathcal{D}}^{\vee} \circ \theta$ is an arrow in $D_{\text{qcoh}}^+(\mathcal{O}_{[\mathcal{W}/T]})$. By the mapping cone axiom of $D_{\text{qcoh}}^+(\mathcal{O}_{[\mathcal{W}/T]})$, we obtain a lift $\phi_{\mathcal{W}}^{\vee}$ that is an arrow in $D_{\text{qcoh}}^+(\mathcal{O}_{[\mathcal{W}/T]})$. This makes $\phi_{\mathcal{W}}^{\vee}$ a T -equivariant obstruction theory of \mathcal{W} .

Next, we show that the cosection $\sigma_{\mathcal{W}/\mathcal{D}}$ lifts to a cosection

$$\sigma_{\mathcal{W}} : \mathcal{O}b_{\mathcal{W}} = H^1(\mathbb{E}_{\mathcal{W}}) \longrightarrow \mathcal{O}_{\mathcal{W}}.$$

Indeed, that $H^1(\mathbb{E}_{\mathcal{W}})$ is the absolute obstruction sheaf of \mathcal{W}/\mathcal{D} , and that $\sigma_{\mathcal{W}/\mathcal{D}}$ lifts to a cosection of $H^1(\mathbb{E}_{\mathcal{W}})$ shown in [CLLL, Lemma 2.10] allow us to define $\sigma_{\mathcal{W}}$ to be this lift, which is T -equivariant since $\sigma_{\mathcal{W}/\mathcal{D}}$ is. Thus the T -equivariant $(\phi_{\mathcal{W}}^{\vee}, \sigma_{\mathcal{W}})$ defines a T -equivariant cosection localized virtual cycle of \mathcal{W} .

To apply virtual localization, we need to verify that for $\mathcal{W}_{(\Gamma)} \subset \mathcal{W}$, which is the image of $\iota_{\Gamma} : \mathcal{W}_{\Gamma} \rightarrow \mathcal{W}$ (cf. Proposition 2.24), we can find a two-term complex of locally free sheaves $[\mathcal{F}_{(\Gamma),0} \rightarrow \mathcal{F}_{(\Gamma),1}]$ on $[\mathcal{W}_{(\Gamma)}/T]$ so that the moving part is

$$(3.5) \quad (\mathbb{E}_{\mathcal{W}}|_{\mathcal{W}_{(\Gamma)}})^{\text{mv}} \cong_{q.i.} [\mathcal{F}_{(\Gamma),0} \rightarrow \mathcal{F}_{(\Gamma),1}].$$

⁴Similar arguments can be found in Section 3. It will also be addressed in [CL2].

First, as $\mathcal{W}_{(\Gamma)} \subset \mathcal{W}^T$ is an open and closed substack, we have the direct sum decomposition as complexes of quasi-coherent sheaves on $[\mathcal{W}_{(\Gamma)}/T]$:

$$(3.6) \quad \mathbb{E}_{\mathcal{W}}|_{\mathcal{W}_{(\Gamma)}} = \bigoplus_k (\mathbb{E}_{\mathcal{W}}|_{\mathcal{W}_{(\Gamma)}})^{(k)},$$

where the superscript (k) means the corresponding weight k part.

Since $\mathbb{E}_{\mathcal{W}}|_{\mathcal{W}_{(\Gamma)}}$ is a complex of sheaves of $\mathcal{O}_{\mathcal{W}_{\Gamma}}$ -modules with coherent cohomologies, the same is true for all $(\mathbb{E}_{\mathcal{W}}|_{\mathcal{W}_{(\Gamma)}})^{(k)}$. By the explicit description of \mathcal{W}_{Γ} , it is direct to check that \mathcal{W}_{Γ} has the resolution property and its coarse moduli is projective. Since $\mathcal{W}_{\Gamma} \rightarrow \mathcal{W}_{(\Gamma)}$ is a finite quotient, the same is true for $\mathcal{W}_{(\Gamma)}$. Thus applying [Kr2, Prop 5.1] and [Huy, Prop 3.5 and Lem 3.6], and using that for any closed $\xi \in \mathcal{W}_{(\Gamma)}$, $H^i(\mathbb{E}_{\mathcal{W}}|_{\xi}) = 0$ for $i \notin [0, 1]$, we conclude that each $(\mathbb{E}_{\mathcal{W}}|_{\mathcal{W}_{(\Gamma)}})^{(k)} \otimes \mathbf{L}_{-k}$ is quasi-isomorphic to a two-term complex of finite rank locally free sheaves of amplitude $[-1, 0]$ on $\mathcal{W}_{(\Gamma)}$. Finally, since $\mathbb{E}_{\mathcal{W}}|_{\mathcal{W}_{(\Gamma)}}$ has coherent cohomologies, all but finitely many of $(\mathbb{E}_{\mathcal{W}}|_{\mathcal{W}_{(\Gamma)}})^{(k)}$'s are quasi-isomorphic to 0. This verifies the requirement to apply Theorem 3.1. Following the convention, we agree

$$e(N_{(\Gamma)}^{\text{vir}}) = e(\mathcal{F}_{(\Gamma),0})/e(\mathcal{F}_{(\Gamma),1}).$$

Proposition 3.2. *Let $[\mathcal{W}]_{\text{loc}}^{\text{vir}}$ (resp. $[\mathcal{W}_{(\Gamma)}]_{\text{loc}}^{\text{vir}}$) be the cosection localized virtual cycle of $\phi_{\mathcal{W}/\mathcal{D}}^{\vee}$ (resp. of $(\phi_{\mathcal{W}}^{\vee}|_{\mathcal{W}_{(\Gamma)}})^T$), let $j_{(\Gamma)}^- : \mathcal{W}_{(\Gamma)}^- \rightarrow (\mathcal{W}^-)^T$ be the inclusion. Then*

$$(3.7) \quad [\mathcal{W}]_{\text{loc}}^{\text{vir}} = \sum_{\Gamma} j_{(\Gamma)*}^- \left(\frac{[\mathcal{W}_{(\Gamma)}]_{\text{loc}}^{\text{vir}}}{e(N_{(\Gamma)}^{\text{vir}})} \right) \in A_*^T(\mathcal{W}^-)^T[\mathfrak{t}^{-1}].$$

Proof. By our construction of the T -equivariant $(\phi_{\mathcal{W}}^{\vee}, \sigma_{\mathcal{W}})$, we can apply the virtual localization theorem Theorem 3.1 to get (3.7), with $[\mathcal{W}]_{\text{loc}}^{\text{vir}}$ the cosection localized virtual cycle of $(\phi_{\mathcal{W}}^{\vee}, \sigma_{\mathcal{W}})$, and $[\mathcal{W}_{(\Gamma)}]_{\text{loc}}^{\text{vir}}$ as stated in the proposition. To complete the proof of the proposition, we need to verify that the cosection localized virtual cycle of $(\phi_{\mathcal{W}}^{\vee}, \sigma_{\mathcal{W}})$ and that of $(\phi_{\mathcal{W}/\mathcal{D}}^{\vee}, \sigma_{\mathcal{W}/\mathcal{D}})$ are identical. The proof of this is parallel to that of Proposition 3.6, and will be omitted. \square

3.2. The fixed part. In this subsection, we determine the cycle $[\mathcal{W}_{(\Gamma)}]_{\text{loc}}^{\text{vir}}$, using Proposition 2.34.

We continue to denote $\mathcal{W} = \mathcal{W}_{g,\gamma,\mathbf{d}}$. Recall that $\mathcal{W}^- (= \mathcal{W}_{g,\gamma,\mathbf{d}}^- \subset \mathcal{W})$ consists of closed points $\xi \in \mathcal{W}$ such that $(\varphi = 0) \cup (\rho = \sum \varphi_i^5 = 0) = \mathcal{C}$ (cf. [CLLL, Lemma 2.11]). Let

$$\mathcal{W}^{\sim} = \{\xi \in \mathcal{W} \mid (\varphi = 0) \cup (\rho = 0) = \mathcal{C}\}.$$

By [CLLL, Coro 3.23], \mathcal{W}^{\sim} is proper.

Definition 3.3. *We say $\beta \in A_*^T(\mathcal{W}^-)^T$ is weakly zero, denoted by $\beta \sim 0$, if its image in $A_*^T(\mathcal{W}^{\sim})^T$ is zero.*

We quote a vanishing proved in [CL2].

Proposition 3.4 ([CL2]). *In case $\Gamma \in \Delta^{\text{fl}}$ is narrow and irregular, and not a loop, then $[\mathcal{W}_{(\Gamma)}]_{\text{loc}}^{\text{vir}} \sim 0$.*

As a weakly zero class in localization formula can be treated as zero, in using (3.7) to get a numerical relation, we only need to sum over all $\Gamma \in \Delta^{\text{reg}}$.

In the following, we fix a $\Gamma \in \Delta^{\text{reg}}$. As Γ will be fixed throughout the remaining of this subsection, we will skip Γ from the notation $V(\Gamma)$, etc., such as using V and E to denote $V(\Gamma)$ and $E(\Gamma)$ respectively. Accordingly $V^S = V^S(\Gamma)$, and $V_0^{0,1} = V_0^{0,1}(\Gamma)$, etc.. Note that for any $(\xi, \epsilon) \in \mathcal{W}_{\Gamma}$, since Γ is regular, $\Gamma_{\xi} = \Gamma^{\text{fl}}$, thus ϵ gives an isomorphism $\Gamma \cong \Gamma_{\xi}$, and any $a \in E \cup V$ is canonically associated, via ϵ , to an $a \in E(\Gamma_{\xi}) \cup V(\Gamma_{\xi})$. In this way, for $\xi = (\Sigma^{\mathcal{C}}, \mathcal{C}, \dots)$, we can talk about subcurves $\mathcal{C}_a \subset \mathcal{C}$ indexed by $a \in E \cup V$.

Let $v \in V_0^S \cup V_{\infty}^S$. Since \mathcal{W}_v defined before Proposition 2.29 is the moduli of T -equivariant $|v|$ -framed stable MSP fields, applying [CLLL] we obtain its cosection localized virtual cycle, denoted by $[\mathcal{W}_v]_{\text{loc}}^{\text{vir}}$. For $v \in V_1^S$, $\mathcal{W}_v \cong \overline{\mathcal{M}}_{g_v, |E_v \cup S_v|}$, thus we set $[\mathcal{W}_v]_{\text{loc}}^{\text{vir}} = [\mathcal{W}_v]$. For unstable $v \in V_0 - V_0^S$, let $[\mathcal{W}_v]_{\text{loc}}^{\text{vir}} = -[Q_5]$, where $Q_5 \subset \mathbb{P}^4$ is the Fermat quintic. We continue to use $\iota_{\Gamma} : \mathcal{W}_{\Gamma} \rightarrow \mathcal{W}_{(\Gamma)}$ (cf. (2.11)) and use $\iota_{\Gamma}^{-} : \mathcal{W}_{\Gamma}^{-} \rightarrow \mathcal{W}_{(\Gamma)}^{-}$ induced by ι_{Γ} .

Proposition 3.5. *Let $\Gamma \in \Delta^{\text{reg}}$. Then*

$$[\mathcal{W}_{(\Gamma)}]_{\text{loc}}^{\text{vir}} = \frac{1}{|\text{Aut}(\Gamma)|} \frac{1}{\prod_{e \in E_0} d_e} \frac{1}{\prod_{e \in E_{\infty}} |G_e|} (\iota_{\Gamma}^{-})_* \left(\prod_{v \in V_0 \cup V_1^S \cup V_{\infty}^S} [\mathcal{W}_v]_{\text{loc}}^{\text{vir}} \right).$$

The remainder of this subsection is devoted to prove this proposition. Let $\Gamma \in \Delta^{\text{fl}}$, not necessarily regular. We first reconstruct $[\mathcal{W}_{\Gamma}]_{\text{loc}}^{\text{vir}}$. Let \mathcal{D}_{Γ} be the smooth Artin stack of Γ -framed curves $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, \mathcal{N})$ as defined in and after Definition 2.25. Let

$$j : \mathcal{D}_{\Gamma} \longrightarrow \mathcal{D} \quad \text{and} \quad q_{\Gamma} : \mathcal{W}_{\Gamma} \longrightarrow \mathcal{D}_{\Gamma}$$

Then we have the Cartesian square via the tautological morphisms

$$(3.8) \quad \begin{array}{ccc} \mathcal{W}_{\Gamma} & \xrightarrow{\iota_{\Gamma}} & \mathcal{W} \\ \downarrow q_{\Gamma} & & \downarrow q \\ \mathcal{D}_{\Gamma} & \xrightarrow{\zeta_{\Gamma}} & \mathcal{D}. \end{array}$$

Let $(\Sigma^{\mathcal{C}_{\Gamma}}, \mathcal{C}_{\Gamma}, \mathcal{L}_{\Gamma}, \mathcal{N}_{\Gamma}, \dots)$ with $\pi_{\Gamma} : \mathcal{C}_{\Gamma} \rightarrow \mathcal{W}_{\Gamma}$ be the universal family of \mathcal{W}_{Γ} . Let \mathcal{P}_{Γ} and \mathcal{V}_{Γ} be defined as in (3.3) with \mathcal{L} , etc., replaced by \mathcal{L}_{Γ} , etc.. Parallel to the construction of the relative obstruction theory $\phi_{\mathcal{W}/\mathcal{D}}^{\vee}$ of $\mathcal{W} \rightarrow \mathcal{D}$, we obtain a T -equivariant relative obstruction theory

$$(3.9) \quad \phi_{\mathcal{W}_{\Gamma}/\mathcal{D}_{\Gamma}}^{\vee} : \mathbb{T}_{\mathcal{W}_{\Gamma}/\mathcal{D}_{\Gamma}} \longrightarrow \mathbb{E}_{\mathcal{W}_{\Gamma}/\mathcal{D}_{\Gamma}} = (R\pi_{\Gamma*} \mathcal{V}_{\Gamma})^T.$$

We will show later that

$$(3.10) \quad \mathcal{O}b_{\mathcal{W}_\Gamma} = \text{coker}\{q_\Gamma^* \mathbb{T}_{\mathcal{D}_\Gamma}[-1] \longrightarrow H^1(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma})\} = (\iota_\Gamma^* \mathcal{O}b_{\mathcal{W}})^T.$$

Granting this, $(\iota_\Gamma^* \sigma_{\mathcal{W}})^T$ induces a cosection of $\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$, liftable to a cosection of its absolute obstruction sheaf.

Proposition 3.6. *The cosection localized T -equivariant cycle $[\mathcal{W}_\Gamma]_{\text{loc}}^{\text{vir}}$ using $(\phi_{\mathcal{W}_\Gamma/\mathcal{D}_T}^\vee, \sigma_{\mathcal{W}_\Gamma/\mathcal{D}_T})$ and using $((\iota_\Gamma^* \phi_{\mathcal{W}}^\vee)^T, (\iota_\Gamma^* \sigma_{\mathcal{W}})^T)$ are identical.*

Here, an object of \mathcal{D}_T is an object in \mathcal{D} with a T -action. Before proving this proposition, we need

Lemma 3.7. *The tautological map $q_\Gamma^* \mathbb{T}_{\mathcal{D}_\Gamma} \rightarrow (\iota_\Gamma^* q^* \mathbb{T}_{\mathcal{D}})^T$ is an isomorphism in the derived category $D_{\text{qcoh}}^+(\mathcal{O}_{\mathcal{W}_\Gamma})$.*

Proof. For notational simplicity, we study a simpler problem. Let \mathcal{D}' be the stack of triples $(\Sigma^c, \mathcal{C}, \mathcal{L})$ of pointed twisted curves with invertible sheaves, let \mathcal{D}'_T be the stack of T -equivariant such triples $(\Sigma^c, \mathcal{C}, \mathcal{L})$, and let $\zeta' : \mathcal{D}'_T \rightarrow \mathcal{D}'$ be the forgetful morphism, forgetting the T -equivariant structure. Then a simple argument shows that the Lemma follows from the claim that $\zeta'^* \mathbb{T}_{\mathcal{D}'}$ is canonically a T -complex, and the tautological map $\mathbb{T}_{\mathcal{D}'_T} \rightarrow \zeta'^* \mathbb{T}_{\mathcal{D}'}$ factors through a T -equivariant isomorphism

$$(3.11) \quad \mathbb{T}_{\mathcal{D}'_T} \xrightarrow{\cong} (\zeta'^* \mathbb{T}_{\mathcal{D}'})^T.$$

To prove the claim, we transform a triple $(\Sigma^c, \mathcal{C}, \mathcal{L})$ to a \mathbb{C}^* -pair $\mathcal{R} \subset \mathcal{S}$:

$$\mathcal{S} = \mathbb{P}_{\mathcal{C}}(\mathcal{L} \oplus 1), \quad \mathcal{R} = \mathcal{S} \times_{\mathcal{C}} \Sigma^c \subset \mathcal{S} \quad \text{and} \quad t \cdot [a, 1] = [ta, 1], \quad t \in \mathbb{C}^*.$$

Let \mathcal{S} be the stack of such \mathbb{C}^* -pairs $\mathcal{R} \subset \mathcal{S}$. Then the mentioned construction defines an equivalence of stacks $\mathcal{D}' \cong \mathcal{S}$. Furthermore, if we let \mathcal{S}_T be the stack of T -equivariant \mathbb{C}^* -pairs in \mathcal{S} , then we have an equivalence of stacks $\mathcal{D}'_T \cong \mathcal{S}_T$. Thus we have induced isomorphisms $\mathbb{T}_{\mathcal{D}'} \cong \mathbb{T}_{\mathcal{S}}$ and $\mathbb{T}_{\mathcal{D}'_T} \cong \mathbb{T}_{\mathcal{S}_T}$. Let $\eta : \mathcal{S}_T \rightarrow \mathcal{S}$ be the forgetful morphism. Since $\eta^* \mathbb{T}_{\mathcal{S}}$ has a canonical T -action, the isomorphism $\eta^* \mathbb{T}_{\mathcal{S}} \cong \zeta'^* \mathbb{T}_{\mathcal{D}'}$ provides the latter a T -action. Furthermore, as $\mathbb{T}_{\mathcal{S}_T} \rightarrow \eta^* \mathbb{T}_{\mathcal{S}}$ factors through an isomorphism $\mathbb{T}_{\mathcal{S}_T} \rightarrow (\eta^* \mathbb{T}_{\mathcal{S}})^T$, $\mathbb{T}_{\mathcal{D}'_T} \rightarrow \zeta'^* \mathbb{T}_{\mathcal{D}'}$ factors through a T -equivariant isomorphism (3.11). This proves the lemma. \square

Proof of Proposition 3.6. We apply the construction of relative obstruction theory in [CL1] to the square (3.8) to obtain the following commutative diagrams

$$(3.12) \quad \begin{array}{ccccc} \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \xrightarrow{=} & (\iota_\Gamma^* \mathbb{E}_{\mathcal{W}/\mathcal{D}})^T & \xrightarrow{\text{inj}} & \iota_\Gamma^* \mathbb{E}_{\mathcal{W}/\mathcal{D}} \\ \uparrow \phi_{\mathcal{W}_\Gamma/\mathcal{D}_T}^\vee & & \uparrow (\phi_{\mathcal{W}/\mathcal{D}}^\vee)^T & & \uparrow \phi_{\mathcal{W}/\mathcal{D}}^\vee \\ \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \xrightarrow{\mathbf{g}_1} & (\iota_\Gamma^* \mathbb{T}_{\mathcal{W}/\mathcal{D}})^T & \xrightarrow{\text{inj}} & \iota_\Gamma^* \mathbb{T}_{\mathcal{W}/\mathcal{D}}, \end{array}$$

where the upper-left arrow is an isomorphism, which follows from the definition of $\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$. We then take the T -invariant part of (3.4) to get the upper two lines of the following commutative diagram in $D_{qcoh}^+(\mathcal{O}_{\mathcal{W}_\Gamma})$, where the lower two lines are induced by the square (3.8), and the lower-left isomorphism is due to Lemma 3.7:

$$\begin{array}{ccccccc}
 & \longrightarrow & (\iota_\Gamma^* q^* \mathbb{T}_{\mathcal{D}})^T[-1] & \xrightarrow{\tau} & (\iota_\Gamma^* \mathbb{E}_{\mathcal{W}/\mathcal{D}})^T & \longrightarrow & (\iota_\Gamma^* \mathbb{E}_{\mathcal{W}})^T \xrightarrow{+1} \longrightarrow \\
 & & \parallel & & \uparrow (\phi_{\mathcal{W}/\mathcal{D}}^\vee)^T & & \uparrow \\
 (3.13) & \longrightarrow & (\iota_\Gamma^* q^* \mathbb{T}_{\mathcal{D}})^T[-1] & \longrightarrow & (\iota_\Gamma^* \mathbb{T}_{\mathcal{W}/\mathcal{D}})^T & \longrightarrow & (\iota_\Gamma^* \mathbb{T}_{\mathcal{W}})^T \xrightarrow{+1} \longrightarrow \\
 & & \uparrow \cong & & \uparrow \mathfrak{g}_1 & & \uparrow \\
 & \longrightarrow & (q_\Gamma^* \mathbb{T}_{\mathcal{D}_\Gamma})[-1] & \longrightarrow & \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \longrightarrow & \mathbb{T}_{\mathcal{W}_\Gamma} \xrightarrow{+1} \longrightarrow .
 \end{array}$$

By (3.12), the composition of the middle vertical arrows is $\phi_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}^\vee$. As $\mathbb{T}_{\mathcal{D}_\Gamma}$ is locally perfect of amplitude $[-1, 0]$, we have an exact sequence of cone-stacks induced from the upper and lower lines of (3.13) and [BF, Prop 2.7]:

$$\begin{array}{ccccc}
 h^1/h^0(q_\Gamma^* \mathbb{T}_{\mathcal{D}_\Gamma}[-1]) & \longrightarrow & h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}) & \xrightarrow{\lambda} & h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma}) \\
 (3.14) \quad \uparrow \cong & & \uparrow (\phi_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}^\vee)^* & & \uparrow (\phi_{\mathcal{W}_\Gamma}^\vee)^* \\
 h^1/h^0(q_\Gamma^* \mathbb{T}_{\mathcal{D}_\Gamma}[-1]) & \longrightarrow & h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}) & \longrightarrow & h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma}).
 \end{array}$$

Applying the argument analogs to [CL1, Coro 2.9] (also see [KKP, Prop 3]), we conclude that the two intrinsic normal cones

$$\mathfrak{C}_{\mathcal{W}_\Gamma} \subset h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma}) \quad \text{and} \quad \mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} \subset h^1/h^0(\mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma})$$

satisfy

$$\lambda^*(\mathfrak{C}_{\mathcal{W}_\Gamma}) = \mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}.$$

Since the two cosections of $\mathcal{O}b_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$ and of $\mathcal{O}b_{\mathcal{W}_\Gamma}$ are compatible under λ , we conclude that the localized virtual classes using the cone $\mathfrak{C}_{\mathcal{W}_\Gamma}$ and using the cone $\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$ are identical (also see [KKP, Prop 3]). This proves that the the cycles $[\mathcal{W}_\Gamma]_{\text{loc}}^{\text{vir}}$ defined using $\phi_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}^\vee$ and using $(\iota_\Gamma^* \phi_{\mathcal{W}})^T$ are identical. \square

We need a few lemmas before proving Proposition 3.5. Let $a \in N(\Gamma)^*$ and let Γ' be the decoupling of Γ along a . Let $\phi : \mathcal{W}_\Gamma \rightarrow \mathcal{W}_{\Gamma'}$ be the isomorphism stated in (2.12). Let S be any scheme and let $\xi \in \mathcal{W}_\Gamma(S)$ be any family with $\pi : \mathcal{C} \rightarrow S$ its domain and \mathcal{L}, \mathcal{N} its associated sheaves. Let $\xi' \in \mathcal{W}_{\Gamma'}(S)$ be the image of ξ under ϕ , of domain $\pi' : \mathcal{C}' \rightarrow S$ and sheaves \mathcal{L}' and \mathcal{N}' . Note that \mathcal{C}' is derived from \mathcal{C} by partial resolution along the S -family of nodes $\mathcal{Q}_a \subset \mathcal{C}$. (Namely, there is an canonical finite S -morphism $\eta : \mathcal{C}' \rightarrow \mathcal{C}$ that is an isomorphism away from $\eta^{-1}(\mathcal{Q}_a)$, and $\eta^{-1}(\mathcal{Q}_a) \rightarrow \mathcal{Q}$ is a two-to-one morphism.)

As in (3.2), we set

$$\mathcal{L}^{\log} = \mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathcal{C}}) \quad \text{and} \quad \mathcal{P}^{\log} = \mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}/S}^{\log}(-\Sigma_{(1,\rho)}^{\mathcal{C}}).$$

And same for \mathcal{L}'^{\log} and \mathcal{P}'^{\log} on \mathcal{C}' .

Lemma 3.8. *Let the notations be as stated. Then we have*

- (1) *a quasi-isomorphism: $(R\pi_* \mathcal{L}^{\log})^T \cong (R\pi'_* \mathcal{L}'^{\log})^T$;*
- (2) *a distinguished triangle:*

$$(R\pi_* \mathcal{P}^{\log})^T \longrightarrow (R\pi'_* \mathcal{P}'^{\log})^T \longrightarrow (R\pi_*(\mathcal{P}^{\log}|_{\mathcal{Q}_a}))^T \xrightarrow{+1}.$$

Proof. We first prove the case where $a = (e, v)$ such that $v \in V_{\infty}^S(\Gamma)$. Let $\eta : \mathcal{C}' \rightarrow S$ be the tautological morphism as before. Using that the two new markings of \mathcal{C}' are both labeled by $(1, \varphi)$, we have the tautological exact sequence of $\mathcal{O}_{\mathcal{C}'}$ -modules:

$$0 \longrightarrow \mathcal{P}^{\log} \longrightarrow \eta_*(\mathcal{P}'^{\log}) \longrightarrow \mathcal{P}^{\log}|_{\mathcal{Q}_a} \longrightarrow 0.$$

Applying $R\pi_*$ we obtain (2).

We now prove (1) in this case. Without loss of generality, we can assume Γ is connected. In case Γ' is connected, let $a' = (e, v') \in N(\Gamma')$, where $v' \in V_1(\Gamma)$ is the other vertex attached to e . Let $\mathcal{Q}_{a'} \subset \mathcal{C}'$ be the associated S -family of nodes, and decompose Γ' along $\mathcal{Q}_{a'}$ to obtain $\mathcal{C}'' \cup \mathcal{C}_e$, which is a disjoint union of two S -families of connected curves, indexed like before.

In case Γ' is disconnected, let $\Gamma' = \Gamma'' \cup |e|$, where $|e|$ is the connected component containing the edge e . We decompose $\mathcal{C}' = \mathcal{C}'' \cup \mathcal{C}_e$ into S -families of connected curves, accordingly. We agree $\mathcal{Q}_{a'} = \emptyset$ in this case.

In either case, both \mathcal{C}'' and \mathcal{C}_e are S -subfamilies of \mathcal{C} . Let $\iota : \mathcal{C}'' \rightarrow \mathcal{C}$ and $\iota' : \mathcal{C}' \rightarrow \mathcal{C}'$ be the tautological embeddings. Then we have the obvious exact sequence of $\mathcal{O}_{\mathcal{C}}$ -modules

$$0 \longrightarrow \iota_*(\mathcal{L}''^{\log}(-\mathcal{Q}_{a'})) \longrightarrow \mathcal{L}^{\log} \longrightarrow \mathcal{L}^{\log}|_{\mathcal{C}_e} \longrightarrow 0,$$

and the obvious exact sequence of $\mathcal{O}_{\mathcal{C}'}$ -modules

$$0 \longrightarrow \iota'_*(\mathcal{L}''^{\log}(-\mathcal{Q}_{a'})) \longrightarrow \mathcal{L}'^{\log} \longrightarrow \mathcal{L}'^{\log}|_{\mathcal{C}_e} \longrightarrow 0.$$

Here we used the fact that one of the two new markings in \mathcal{C}'' is labeled by $(1, \varphi)$ and the other labeled by $(1, \rho)$. We claim $R\pi_*(\mathcal{L}^{\log}|_{\mathcal{C}_e}) = 0$. In fact, we first prove the case where S is a point. As $\mathcal{L}^{\log}|_{\mathcal{C}_e} = \mathcal{L}|_{\mathcal{C}_e}$ has negative degree, we have $H^0(\mathcal{L}^{\log}|_{\mathcal{C}_e}) = 0$. We then use Serre-duality

$$H^1(\mathcal{L}^{\log}|_{\mathcal{C}_e}) = H^1(\mathcal{L}|_{\mathcal{C}_e}) = H^0(\mathcal{L}^{\vee}|_{\mathcal{C}_e} \otimes \omega_{\mathcal{C}_e}(\mathcal{Q}_a)(-\mathcal{Q}_a))^{\vee}.$$

Since T acts on $(\mathcal{L}^{\vee}|_{\mathcal{C}_e} \otimes \omega_{\mathcal{C}_e}(\mathcal{Q}_a))|_{\mathcal{Q}_a}$ trivially, we obtain $H^i(\mathcal{L}^{\log}|_{\mathcal{C}_e})^T = 0$, by Lemma 2.3. Therefore, applying base change property, we have proven the claim $R\pi_*(\mathcal{L}^{\log}|_{\mathcal{C}_e}) = 0$. Thus we get (1).

The case $v \in V_{\infty}^{0,2}(\Gamma)$ is the same. The case $v \in V_1(\Gamma)$ is simpler, using that since the T -action on $\mathcal{L}|_{\mathcal{Q}_a}$ is via scaling by t , we have $(R\pi_*(\mathcal{L}^{\log}|_{\mathcal{Q}_a}))^T$ and $(R\pi_*(\mathcal{P}^{\log}|_{\mathcal{Q}_a}))^T$ are quasi-isomorphic to 0. \square

We continue to denote by Γ' the decoupling of Γ along $a \in N(\Gamma)^*$. We look at the tautological morphism $\phi : \mathcal{W}_\Gamma \rightarrow \mathcal{W}_{\Gamma'}$ in (2.12).

Lemma 3.9. *We have the identity $\phi^*[\mathcal{W}_{\Gamma'}]_{\text{loc}}^{\text{vir}} = [\mathcal{W}_\Gamma]_{\text{loc}}^{\text{vir}}$.*

Proof. As before, we let $(\mathcal{C}_\Gamma, \mathcal{L}_\Gamma, \mathcal{N}_\Gamma, \dots)$ with $\pi_\Gamma : \mathcal{C}_\Gamma \rightarrow \mathcal{W}_\Gamma$ be the universal family of \mathcal{W}_Γ . (Same for $\mathcal{W}_{\Gamma'}$; although Γ' is possibly disconnected, the study for \mathcal{W}_Γ applied to $\mathcal{W}_{\Gamma'}$ as well.) By our construction of decoupling, we have canonical

$$\tilde{\phi} : \phi^* \mathcal{C}_{\Gamma'} \longrightarrow \mathcal{C}_\Gamma, \quad \text{and} \quad \tilde{\phi}^* \mathcal{L}_{\Gamma'} \xrightarrow{\cong} \mathcal{L}_\Gamma, \quad \tilde{\phi}^* \mathcal{N}_{\Gamma'} \xrightarrow{\cong} \mathcal{N}_\Gamma.$$

As $a = (e, v)$ with $v \in V_\infty^S(\Gamma)$ (cf. proof of Lemma 3.8), we see that $\mathcal{N}_\Gamma|_{\mathcal{Q}_a}$ has non-trivial weight; thus $(R\pi_{\Gamma*} \mathcal{N}_\Gamma|_{\mathcal{Q}_a})^T = 0$. Let \mathcal{V}_Γ be as defined before (3.9) (cf. (3.3)). Applying Lemma 3.8 and the previous discussion, we obtain the d.t.

$$(3.15) \quad (R\pi_{\Gamma*} \mathcal{V}_\Gamma)^T \longrightarrow (\phi^* R\pi_{\Gamma'*} \mathcal{V}_{\Gamma'})^T \longrightarrow (R\pi_{\Gamma*} \mathcal{V}_\Gamma^c|_{\mathcal{Q}_a})^T \xrightarrow{+1},$$

where $\mathcal{V}_\Gamma^c = \mathcal{P}_\Gamma^{\text{log}} \oplus \mathcal{L}_\Gamma \otimes \mathcal{N}_\Gamma \otimes \mathbf{L}_1$. On the other hand, using the non-vanishing of ρ and ν_1 along \mathcal{Q}_a , we have $(R\pi_{\Gamma*} \mathcal{V}_\Gamma^c|_{\mathcal{Q}_a})^T \cong \mathcal{O}_{\mathcal{W}_\Gamma}^{\oplus 2}$.

Let \mathcal{D}_Γ be the stack of Γ -framed curves defined in and after Definition 2.25. As we need to do induction on decoupling, Γ is possibly not connected. Therefore we need to consider the case of not necessarily connected curves. Obviously, we have the forgetting morphism $q_\Gamma : \mathcal{W}_\Gamma \rightarrow \mathcal{D}_\Gamma$ which composed with the forgetful $\mathcal{D}_\Gamma \rightarrow \mathcal{D}_T$ gives us the forgetful $\mathcal{W}_\Gamma \rightarrow \mathcal{D}_T$. We comment that $\mathcal{D}_\Gamma \rightarrow \mathcal{D}_T$ is étale.

We show that there is a tautological morphism $\mathcal{D}_\Gamma \rightarrow \mathcal{D}_{\Gamma'}$ that fits into the commutative diagram of morphisms

$$(3.16) \quad \begin{array}{ccc} \mathcal{W}_\Gamma & \xrightarrow{\phi} & \mathcal{W}_{\Gamma'} \\ \downarrow q_\Gamma & & \downarrow q_{\Gamma'} \\ \mathcal{D}_\Gamma & \longrightarrow & \mathcal{D}_{\Gamma'}. \end{array}$$

Given a Γ -framed curve $(\mathcal{C}, \Sigma^\mathcal{C}, \mathcal{L}, \mathcal{N})$, following Definition 2.25, \mathcal{C} has a distinguished (T -unbalanced) node q_a , and the two branches of q_a are also labeled by data in Γ . Let $\pi : \mathcal{C}' \rightarrow \mathcal{C}$ be the partial normalization of \mathcal{C} along q_a (and q_a only), and let $\Sigma^{\mathcal{C}'} = \pi^{-1}(\Sigma^\mathcal{C}) \cup \pi^{-1}(q_a)$, where we label points $\pi^{-1}(\Sigma^\mathcal{C})$ by the legs of Γ' that come from legs in Γ and label the two points in $\pi^{-1}(q_a)$ according to which branch its image (under π) lies in. As T -unbalanced nodes remain nodes in any family of T -equivariant curves, this association defines a morphism $\mathcal{D}_\Gamma \rightarrow \mathcal{D}_{\Gamma'}$ that fits into the commutative square (3.16).

Without lose of generality, we can assume Γ is connected. Then Γ' either is connected or has two connected components. We first consider the case where Γ' is connected. Then the lower horizontal line in (3.16) is a $(\mathbb{C}^*)^2$ -torsor,

where the group $(\mathbb{C}^*)^2$ acts on \mathcal{D}_Γ by scaling the gluing of the pullbacks of $\mathcal{L}_{\Gamma'}$ and $\mathcal{N}_{\Gamma'}$ at nodes $\mathcal{Q}_a \subset \mathcal{C}_\Gamma$. Therefore, $\mathcal{D}_\Gamma/\mathcal{D}_{\Gamma'}$ is smooth, and

$$(3.17) \quad q_\Gamma^* \mathbb{T}_{\mathcal{D}_\Gamma/\mathcal{D}_{\Gamma'}} \cong \pi_{\Gamma*}(\mathcal{O}_{\mathcal{C}_\Gamma}^{\oplus 2}|_{\mathcal{Q}_a}) = \mathcal{O}_{\mathcal{W}_\Gamma}^{\oplus 2}.$$

On the other hand, as $\mathcal{W}_{\Gamma'}$ is defined similar to \mathcal{W}_Γ , it comes with a tautological relative obstruction theory which takes the form

$$\phi_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma'}}^\vee : \mathbb{T}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma'}} \longrightarrow \mathbb{E}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma'}} = (R\pi_{\Gamma'*} \mathcal{V}_{\Gamma'})^T.$$

We then form a morphism of d.t.s

$$\begin{array}{ccccc} \mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \xrightarrow{\gamma_1} & \phi^* \mathbb{E}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma'}} & \longrightarrow & (\pi_{\Gamma*} \mathcal{V}_\Gamma^c|_{\mathcal{Q}_a})^T \xrightarrow{+1} \\ \uparrow \phi_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}^\vee & & \uparrow \phi^*(\phi_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma'}}^\vee) & & \uparrow \alpha \\ \mathbb{T}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} & \xrightarrow{\gamma_2} & \phi^* \mathbb{T}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma'}} & \longrightarrow & q_\Gamma^* \mathbb{T}_{\mathcal{D}_\Gamma/\mathcal{D}_{\Gamma'}} \xrightarrow{+1} . \end{array}$$

Here the top d.t. is (3.15), using that $\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma} = (R\pi_{\Gamma*} \mathcal{V}_\Gamma)^T$, etc.; the lower d.t. is from the functoriality of cotangent complexes using the arrows in (3.16). Because ϕ in (3.16) is an isomorphism, the construction of obstruction theories in [CL1] applied to this case ensures that the left square is commutative. Let α be so that it makes the above diagram a morphism between d.t.s.

We claim that α is an isomorphism. Indeed, by the discussion after (3.15), and by the isomorphism (3.17), both $q_\Gamma^* \mathbb{T}_{\mathcal{D}_\Gamma/\mathcal{D}_{\Gamma'}}$ and $(\pi_{\Gamma*} \mathcal{V}_\Gamma^c|_{\mathcal{Q}_a})^T$ are isomorphic to $\mathcal{O}_{\mathcal{W}_\Gamma}^{\oplus 2}$. By the construction of γ_1 and the explicit knowledge of ϕ in (3.16), we conclude that

$$\text{coker } H^0(\gamma_1) \cong \text{coker } H^0(\gamma_2) \quad \text{and} \quad \ker H^1(\gamma_1) \cong \ker H^1(\gamma_2).$$

This implies that α is an isomorphism.

Finally, we check that the T -invariant cosection $\sigma_{\mathcal{W}}$ of the obstruction sheaf of \mathcal{W} lifts to a cosection σ_Γ of the obstruction sheaf of \mathcal{W}_Γ , identical to that using the formula (2.9) in [CLLL]. Let $\sigma_{\Gamma'}$ be the cosection of the obstruction sheaf of $\mathcal{W}_{\Gamma'}$, defined similarly. Then it is straightforward to see that σ_Γ and $\sigma_{\Gamma'}$ are consistent under α .

Since $\mathbb{T}_{\mathcal{D}_\Gamma/\mathcal{D}_{\Gamma'}}$ is a locally free sheaf,

$$h^1/h^0(\mathbb{E}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}) \xrightarrow{h^1/h^0(\gamma_1)} h^1/h^0(\mathbb{E}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma'}})$$

is a smooth quotient. Adding the commutative diagram above, we conclude that the intrinsic normal cone $\mathfrak{C}_{\mathcal{W}_\Gamma/\mathcal{D}_\Gamma}$ is the pullback of the intrinsic normal cone $\mathfrak{C}_{\mathcal{W}_{\Gamma'}/\mathcal{D}_{\Gamma'}}$. Therefore, by the comparison of the two cosections σ_Γ and $\sigma_{\Gamma'}$, we prove the proposition. \square

Proof of Proposition 3.5. Note that Γ in this case is regular. Namely, we are only decoupling either stacky nodes or nodes in \mathbb{C}_1 .

We first work out $[\mathcal{W}_{[v]}]_{\text{loc}}^{\text{vir}}$ for $[v] \in V_{[0]}$. We prove that for $v \in V_0 - V_0^S$,

$$(3.18) \quad [\mathcal{W}_{[v]}]_{\text{loc}}^{\text{vir}} = \frac{1}{\prod_{e \in E_v} d_e} [-Q_5].$$

We look at the case where $E_v = \{e, e'\}$. Let $(\mathcal{C}_{[v]}, \mathcal{L}_{[v]})$ be the part of the universal family over $\mathcal{W}_{[v]}$. Then it is the union of two subfamilies \mathcal{C}_e and $\mathcal{C}_{e'} \subset \mathcal{C}_{[v]}$ with $\mathcal{C}_e \cap \mathcal{C}_{e'} = \mathcal{C}_v$ being the section of nodes of $\mathcal{C}_{[v]}$. Let $\mathcal{L}_v = \mathcal{L}_{[v]}|_{\mathcal{C}_v}$, then the sections $(\varphi_{[v],1}, \dots, \varphi_{[v],5})|_{\mathcal{C}_v}$ define a morphism

$$(3.19) \quad \phi_v : \mathcal{C}_v \rightarrow \mathbb{P}^4$$

Following the proof of Lemma 2.30, we have $\phi_v^* \mathcal{O}_{\mathbb{P}^4}(1) \cong \mathcal{L}_v$. Next, by (2) of Lemma 3.8, assuming $\Sigma_{(1,\rho)}^{\mathcal{C}}$ is empty without loss of generality, we get

$$\left(R^1 \pi_{[v]*} (\mathcal{L}_{[v]}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}_{[v]}/\mathcal{W}_{[v]}}^{\log}) \right)^T \cong \mathcal{L}_v^{\vee \otimes 5} \cong \phi_v^* \mathcal{O}_{\mathbb{P}^4}(-5).$$

Thus $\mathcal{W}_{[v]}$ has the obstruction sheaf isomorphic to $\phi_v^* \mathcal{O}_{\mathbb{P}^4}(-5)$. Furthermore, by Lemma 2.30,

$$\mathcal{W}_{[v]} = {}^{d_e}\sqrt{\mathcal{O}_{\mathbb{P}^4}(1)/\mathbb{P}^4} \times_{\mathbb{P}^4} {}^{d_{e'}}\sqrt{\mathcal{O}_{\mathbb{P}^4}(1)/\mathbb{P}^4}.$$

This proves (3.18). The proofs for other unstable $v \in V_0$ are similar.

Finally, a parallel argument shows that in case $v \in V_0^S$ and $E_v = \{e_1, \dots, e_k\}$, then $\mathcal{W}_{[v]}$ is a $\prod_{i=1}^k \mu_{d_{e_i}}$ -gerbe over $\mathcal{W}_v \cong \overline{\mathcal{M}}_{g_v, n_v}(\mathbb{P}^4, d_v)^p$. This proves

$$(3.20) \quad [\mathcal{W}_{[v]}]_{\text{loc}}^{\text{vir}} = \left(\prod_{i=1}^k d_{e_i} \right)^{-1} \cdot [\mathcal{W}_v]_{\text{loc}}^{\text{vir}}.$$

Finally, applying Lemma 3.9 repeatedly, we obtain $[\mathcal{W}_{\Gamma}]_{\text{loc}}^{\text{vir}} = [\mathcal{W}_{\Gamma^{\text{de}}}]_{\text{loc}}^{\text{vir}}$. Since Γ is regular, the proposition follows from the previous arguments and (3.18). \square

4. VIRTUAL LOCALIZATION, PART 2

In this section, we fix a $\Gamma \in \Delta^{\text{reg}}$ and will work out the contribution to the localization formula from the moving parts associated to Γ . We continue to abbreviate $\mathcal{W} = \mathcal{W}_{g,\gamma,\mathbf{d}}$. Let $\xi \in \mathcal{W}_{\Gamma}$, thus $\mathcal{C} = \mathcal{C}_{\Gamma} \times_{\mathcal{W}} \xi$, etc.. Denote $\mathcal{V} = \mathcal{V}_{\Gamma} \otimes_{\mathcal{O}_{\mathcal{C}_{\Gamma}}} \mathcal{O}_{\mathcal{C}}$. We form

$$\begin{aligned} B_1 &= \text{Aut}(\Sigma^{\mathcal{C}} \subset \mathcal{C}) = \text{Ext}^0(\Omega_{\mathcal{C}}(\Sigma^{\mathcal{C}}), \mathcal{O}_{\mathcal{C}}), \\ B_2 &= \text{Aut}(\mathcal{L}) \oplus \text{Aut}(\mathcal{N}) = H^0(\mathcal{O}_{\mathcal{C}}^{\oplus 2}), \\ B_3 &= \text{Def}(\varphi, \rho, (\nu_1, \nu_2)) = H^0(\mathcal{V}), \\ B_4 &= \text{Def}(\Sigma^{\mathcal{C}} \subset \mathcal{C}) = \text{Ext}^1(\Omega_{\mathcal{C}}(\Sigma^{\mathcal{C}}), \mathcal{O}_{\mathcal{C}}), \\ B_5 &= \text{Def}(\mathcal{L}) \oplus \text{Def}(\mathcal{N}) = H^1(\mathcal{O}_{\mathcal{C}}^{\oplus 2}), \\ B_6 &= \text{Obs}(\varphi, \rho, (\nu_1, \nu_2)) = H^1(\mathcal{V}), \end{aligned}$$

where Aut is the space of infinitesimal automorphisms, etc.. Then all B_i are T -spaces. Let B_i^{mv} be the moving parts of B_i . Then the virtual normal bundle N^{vir} to \mathcal{W}_Γ in \mathcal{W} restricted at ξ is

$$N^{\text{vir}}|_\xi = T_\xi^{\text{mv}} - \text{Ob}_\xi^{\text{mv}} = -B_1^{\text{mv}} - B_2^{\text{mv}} + B_3^{\text{mv}} + B_4^{\text{mv}} + B_5^{\text{mv}} - B_6^{\text{mv}}.$$

It will be clear later that the B_i^{mv} for $\xi \in \mathcal{W}_\Gamma$ forms a vector bundle over \mathcal{W}_Γ . By abuse of notation, we will view B_i^{mv} as such a vector bundle. Then T -equivariant Euler class $e_T(N^{\text{vir}})$ of the virtual normal bundle is given by

$$(4.1) \quad \frac{1}{e_T(N^{\text{vir}})} = \frac{e_T(B_1^{\text{mv}})e_T(B_2^{\text{mv}})e_T(B_6^{\text{mv}})}{e_T(B_3^{\text{mv}})e_T(B_4^{\text{mv}})e_T(B_5^{\text{mv}})}.$$

The goal of this section is to derive an explicit formula of $\frac{1}{e_T(N^{\text{vir}})}$. We will compute $\frac{e_T(B_1^{\text{mv}})}{e_T(B_4^{\text{mv}})}$ in Section 4.1, and compute $\frac{e_T(B_2^{\text{mv}})e_T(B_6^{\text{mv}})}{e_T(B_3^{\text{mv}})e_T(B_5^{\text{mv}})}$ in Section 4.2. The explicit formula of $\frac{1}{e_T(N^{\text{vir}})}$ will be given in Section 4.3.

4.1. The moving part of deforming $\Sigma^\mathcal{C} \subset \mathcal{C}$. In this subsection, we compute

$$\frac{e_T(B_1^{\text{mv}})}{e_T(B_4^{\text{mv}})}.$$

Recall our convention that $F = F(\Gamma)$, $V^{0,2} = V^{0,2}(\Gamma)$, etc., also the convention on nodes:

$$\forall (e, v) \in F : y_{(e,v)} = \mathcal{C}_v \cap \mathcal{C}_e; \quad \forall v \in V^{0,2} \text{ and } E_v = \{e, e'\} : y_{(e,v)} = \mathcal{C}_e \cap \mathcal{C}_{e'}.$$

Set $F^{0,1} = \{(e, v) \in F : v \in V^{0,1}\}$ and $F^S = \{(e, v) \in F : v \in V^S\}$. Then

$$\begin{aligned} B_1^{\text{mv}} &= \bigoplus_{(e,v) \in F^{0,1}} T_{\mathcal{C}_v} \mathcal{C}_e, \\ B_4^{\text{mv}} &= \bigoplus_{(e,v) \in F^S} T_{y_{(e,v)}} \mathcal{C}_e \otimes T_{y_{(e,v)}} \mathcal{C}_v \oplus \bigoplus_{\substack{v \in V^{0,2} \\ E_v = \{e, e'\}}} T_{y_{(e,v)}} \mathcal{C}_e \otimes T_{y_{(e',v)}} \mathcal{C}_{e'}. \end{aligned}$$

Let $(e, v) \in F$, then $T_{y_{(e,v)}} \mathcal{C}_e$ forms a line bundle over \mathcal{W}_e . In case $v \in V^S$, then $T_{y_{(e,v)}} \mathcal{C}_v$ forms a line bundle over \mathcal{W}_v . By abuse of notations, we will view $T_{y_{(e,v)}} \mathcal{C}_e$, $T_{y_{(e,v)}} \mathcal{C}_v$, and $T_{\mathcal{C}_v} \mathcal{C}_e$ as such line bundles. As T acts trivially on \mathcal{W}_e and \mathcal{W}_v ,

$$H_T^2(\mathcal{W}_e; \mathbb{Q}) = H^2(\mathcal{W}_e; \mathbb{Q}) \oplus \mathbb{Q}\mathfrak{t}, \quad H_T^2(\mathcal{W}_v; \mathbb{Q}) = H^2(\mathcal{W}_v; \mathbb{Q}) \oplus \mathbb{Q}\mathfrak{t}.$$

As T acts trivially on $T_{y_{(e,v)}} \mathcal{C}_v$,

$$e_T(T_{y_{(e,v)}} \mathcal{C}_v) = e(T_{y_{(e,v)}} \mathcal{C}_v) = -\psi_{(e,v)},$$

where $\psi_{(e,v)} \in H^2(\mathcal{W}_v; \mathbb{Q})$ is the ψ -class associated to the pointed curves $y_{(e,v)} \in \mathcal{C}_v$. For any $(e, v) \in F$, let $\mathbf{T}_{(e,v)}$ be the line bundle $\mathbf{T}_{(e,v)} = T_{y_{(e,v)}} \mathcal{C}_e$ over \mathcal{W}_e , let

$$w_{(e,v)} := e_T(\mathbf{T}_{(e,v)}) \in H_T^2(\mathcal{W}_e; \mathbb{Q}).$$

Then

$$\begin{aligned} e_T(B_1^{\text{mv}}) &= \prod_{(e,v) \in F^{0,1}} w_{(e,v)}; \\ e_T(B_4^{\text{mv}}) &= \prod_{(e,v) \in F^S} (w_{(e,v)} - \psi_{(e,v)}) \cdot \prod_{\substack{v \in V^{0,2} \\ E_v = \{e, e'\}}} (w_{(e,v)} + w_{(e',v)}). \end{aligned}$$

Therefore,

$$(4.2) \quad \frac{e_T(B_1^{\text{mv}})}{e_T(B_4^{\text{mv}})} = \prod_{(e,v) \in F^S} \frac{1}{w_{(e,v)} - \psi_{(e,v)}} \cdot \prod_{\substack{v \in V^{0,2} \\ E_v = \{e, e'\}}} \frac{1}{w_{(e,v)} + w_{(e',v)}} \cdot \prod_{(e,v) \in F^{0,1}} w_{(e,v)}.$$

It remains to determine $w_{(e,v)}$ for $(e, v) \in F$. The formula of $w_{(e,v)}$ will be given in Lemma 4.1 below. To state Lemma 4.1, we introduce some definitions.

In case $e \in E_0$, let $\mathcal{E}_e \rightarrow \mathcal{W}_e = \sqrt[d_e]{\mathcal{O}_{\mathbb{P}^4}(1)}/\mathbb{P}^4$ be the tautological line bundle such that $\mathcal{E}_e^{\otimes d_e} = \pi_e^* \mathcal{O}_{\mathbb{P}^4}(1)$, where $\pi_e : \mathcal{W}_e \rightarrow \mathbb{P}^4$ is the projection to its coarse moduli space. Let $h \in H^2(\mathbb{P}^4; \mathbb{Q})$ be the hyperplane class, and let $h_e = \pi_e^* h \in H^2(\mathcal{W}_e; \mathbb{Q})$.

For $e \in E_0$, let $v \in V_0$ and $v' \in V_1$ be the two vertices of e . For $e \in E_\infty$, let $v \in V_\infty$ and $v' \in V_1$ be the two vertices of e . Set

$$(4.3) \quad \delta' = \begin{cases} -1, & v' \in V^{0,1}, \\ 0, & v' \in V \setminus V^{0,1}, \end{cases} \quad \delta = \begin{cases} -1, & v \in V^{0,1}, \\ 0, & v \in V \setminus V^{0,1}. \end{cases}$$

Then

$$\omega_{\mathbb{C}}^{\log} \Big|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathcal{C}_e}(\delta y_{(e,v)} + \delta' y_{(e,v')}).$$

For $e \in E_\infty$, set

$$(4.4) \quad r_e = \begin{cases} 1, & d_e \in \mathbb{Z}, \\ 5, & d_e \notin \mathbb{Z}. \end{cases}$$

Then $\mathcal{C}_e \cong \mathbb{P}(r_e, 1)$. Because Γ is regular, if $v \in V_\infty^S$, then $d_e \neq \mathbb{Z}$ and $r_e = 5$. Note that if $v \in V_\infty^{0,1}$, then $d_e \in \mathbb{Z}$ and $r_e = 1$.

Lemma 4.1. (1) When $v \in V_0$, then $w_{(e,v)} = \frac{h_e + t}{d_e}$ and $w_{(e,v')} = -\frac{h_e + t}{d_e}$.
 (2) When $v \in V_\infty \setminus V_\infty^{0,1}$, then $w_{(e,v)} = \frac{t}{r_e d_e}$ and $w_{(e,v')} = -\frac{t}{d_e}$.
 (3) When $v \in V_\infty^{0,1}$, then $w_{(e,v)} = \frac{5t}{5d_e + 1}$ and $w_{(e,v')} = \frac{-5t}{5d_e + 1}$.

Proof. We begin with the case $e \in E_0$. For \mathcal{E}_e and h_e defined before the statement of the Lemma, we have

$$c_1(\mathcal{E}_e) = \frac{h_e}{d_e} \in H^2(\mathcal{W}_e; \mathbb{Q}),$$

and $\mathbf{T}_{(e,v)} \cong \mathcal{E}_e \otimes \mathbf{L}_{1/d_e} \cong \mathbf{T}_{(e,v')}^\vee$ as T -equivariant line bundles over \mathcal{W}_e , where \mathcal{E}_e is equipped with the trivial T -equivariant structure. Therefore $w_{(e,v)} = e_T(\mathbf{T}_{(e,v)})$ and $w_{(e,v')}$ have the expressions stated in part (1) of the Lemma.

We next consider $e \in E_\infty$. In this case, the coarse moduli of \mathcal{W}_e is a point, so $H_T^2(\mathcal{W}_e; \mathbb{Q}) = \mathbb{Q}\mathbf{t}$. We have

$$w_{(e,v')} = e_T(T_{y_{(e,v')}} \mathcal{C}_e), \quad \text{and} \quad w_{(e,v)} = e_T(T_{y_{(e,v)}} \mathcal{C}_e) = -\frac{w_{(e,v')}}{r_e},$$

where r_e is defined in (4.4). Let $\mathcal{P} = \mathcal{L}^{\vee \otimes 5} \otimes \omega_e^{\log}$. By Remark 2.7,

$$\mathcal{P}|_{y_{(e,v)}} \cong (\mathcal{L} \otimes \mathcal{N} \otimes \mathbf{L}_1)|_{y_{(e,v)}} \cong (\mathcal{L} \otimes \mathcal{N} \otimes \mathbf{L}_1)|_{y_{(e,v')}} \cong \mathcal{N}|_{y_{(e,v')}} \cong \mathbf{L}_0.$$

Then for δ and δ' defined in (4.3),

$$\mathcal{P}|_{\mathcal{C}_e} \cong \mathcal{L}^{\vee \otimes 5}|_{\mathcal{C}_e} \otimes \mathcal{O}_{\mathcal{C}_e}(\delta y_{(e,v)} + \delta' y_{(e,v')}).$$

We have

$$\begin{aligned} e_T(\mathcal{P}|_{y_{(e,v')}}) &= -5e_T(\mathcal{L}|_{y_{(e,v')}}) + \delta' w_{(e,v')} = 5\mathbf{t} + \delta' w_{(e,v')}, \\ e_T(\mathcal{P}|_{y_{(e,v)}}) &= e_T(\mathcal{P}|_{y_{(e,v')}}) + \deg(\mathcal{P}|_{\mathcal{C}_e}) w_{(e,v)} = 0 + (-5d_e + \frac{\delta}{r_e} + \delta') w_{(e,v')}. \end{aligned}$$

Therefore $w_{(e,v')} = \frac{5\mathbf{t}}{-5d_e + \delta/r_e} = \frac{5\mathbf{t}}{-5d_e + \delta}$. (Since $\delta = -1$ implies $d_e \in \mathbb{Z} \Leftrightarrow r_e = 1$.) So when $v \in V_\infty \setminus V_\infty^{0,1}$, then $w_{(e,v')} = -\frac{\mathbf{t}}{d_e}$ and $w_{(e,v)} = \frac{\mathbf{t}}{r_e d_e}$; when $v \in V_\infty^{0,1}$, then $w_{(e,v')} = \frac{-5\mathbf{t}}{5d_e + 1}$ and $w_{(e,v)} = \frac{5\mathbf{t}}{5d_e + 1}$. This proves part (2) and part (3) of the Lemma. \square

For references, if $v \in V_\infty$, then

$$\begin{aligned} e_T(\mathcal{L}|_{y_{(e,v')}}) &= -\mathbf{t}, \quad e_T(\mathcal{N}|_{y_{(e,v')}}) = 0, \quad e_T(\mathcal{P}|_{y_{(e,v')}}) = (5 - \frac{5\delta'}{5d_e - \delta})\mathbf{t}, \\ e_T(\mathcal{L}|_{y_{(e,v)}}) &= \frac{\delta\mathbf{t}}{5d_e - \delta} = \frac{\delta\mathbf{t}}{5d_e + 1}, \quad e_T(\mathcal{N}|_{y_{(e,v)}}) = \frac{-5d_e\mathbf{t}}{5d_e - \delta} = -\mathbf{t} - \frac{\delta\mathbf{t}}{5d_e + 1}, \\ e_T(\mathcal{P}|_{y_{(e,v)}}) &= 0. \end{aligned}$$

4.2. The moving part of deforming $(\mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$. In this subsection, we compute

$$(4.5) \quad \frac{e_T(B_2^{\text{mv}})e_T(B_6^{\text{mv}})}{e_T(B_3^{\text{mv}})e_T(B_5^{\text{mv}})}.$$

Because ξ is T -invariant, the spaces $H^i(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathbb{C}}))$, $H^i(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}}))$, etc., are T -vector spaces. As T acts on the fields via $t \cdot (\varphi, \rho, \nu_1, \nu_2) = (\varphi, \rho, t\nu_1, \nu_2)$, the vector space $H^i(\mathcal{V})$ as T -vector space is

$$H^i(\mathcal{V}) = H^i(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathbb{C}}))^{\oplus 5} \oplus H^i(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})) \oplus H^i(\mathcal{L} \otimes \mathcal{N}) \otimes \mathbf{L}_1 \oplus H^i(\mathcal{N}).$$

To compute (4.5), we need to study the moving parts:

$$(H^0(\mathcal{V}) - H^0(\mathcal{O}_{\mathbb{C}}^{\oplus 2}))^{\text{mv}} \quad \text{and} \quad (H^1(\mathcal{V}) - H^1(\mathcal{O}_{\mathbb{C}}^{\oplus 2}))^{\text{mv}}.$$

To achieve this, we use the long exact sequences of T -representations

$$\longrightarrow H^i(\mathcal{V}) \longrightarrow \bigoplus_{v \in V^S \cup E} H^i(\mathcal{V}|_{\mathbb{C}_v}) \longrightarrow \bigoplus_{a \in F^S \cup V^{0,2}} H^i(\mathcal{V}|_{y_a}) \longrightarrow H^{i+1}(\mathcal{V}) \longrightarrow,$$

and

$$\longrightarrow H^i(\mathcal{O}_{\mathbb{C}}^{\oplus 2}) \longrightarrow \bigoplus_{v \in V^S \cup E} H^i(\mathcal{O}_{\mathbb{C}_v}^{\oplus 2}) \longrightarrow \bigoplus_{a \in F^S \cup V^{0,2}} H^i(\mathcal{O}_{y_a}^{\oplus 2}) \longrightarrow H^{i+1}(\mathcal{O}_{\mathbb{C}}^{\oplus 2}) \longrightarrow.$$

We have

$$(4.6) \quad \frac{e_T(B_2^{\text{mv}})e_T(B_6^{\text{mv}})}{e_T(B_3^{\text{mv}})e_T(B_5^{\text{mv}})} = \prod_{v \in V^S} A'_v \prod_{e \in E} A'_e \prod_{a \in F^S \cup V^{0,2}} A_a$$

where A'_v , A'_e and A_a are contributions from $(H^i(\mathcal{V}|_{\mathbb{C}_v}) - H^i(\mathcal{O}_{\mathbb{C}_v}^{\oplus 2}))^{\text{mv}}$ ($i = 0, 1$), $(H^i(\mathcal{V}|_{\mathbb{C}_e}) - H^i(\mathcal{O}_{\mathbb{C}_e}^{\oplus 2}))^{\text{mv}}$ ($i = 0, 1$), and $(H^0(\mathcal{V}|_{y_a}) - H^0(\mathcal{O}_{y_a}^{\oplus 2}))^{\text{mv}}$ respectively. We will derive explicit formulas of A'_v , A'_e , and A_a in Section 4.2.1, Section 4.2.2, and Section 4.2.3, respectively.

4.2.1. Contribution from stable vertices. We first introduce some notation:

- Given a stable vertex $v \in V^S$, let $\pi_v : \mathcal{C}_v \rightarrow \mathcal{W}_v$ be the universal curve; let \mathcal{L}_v and \mathcal{N}_v be the universal line bundle over \mathcal{C}_v .
- Given $v \in V_0^S$, let $\phi_v : \mathcal{C}_v \rightarrow \mathbb{P}^4$ be defined as in (3.19).
- Given $v \in V_1^S$, let $\mathbb{E} := \pi_* \omega_{\pi_v}$ be the Hodge bundle, where $\omega_{\pi_v} \rightarrow \mathcal{C}_v$ is the relative dualizing sheaf. Then $\mathbb{E}^\vee = R^1 \pi_{v*} \mathcal{O}_{\mathcal{C}_v}$.

The contribution A'_v from a stable vertex $v \in V^S$ is given by the following lemma.

Lemma 4.2.

$$A'_v = \begin{cases} \frac{1}{e_T(R\pi_{v*} \phi_v^* \mathcal{O}_{\mathbb{P}^4}(1) \otimes \mathbf{L}_1)}, & v \in V_0^S; \\ \left(\frac{e_T(\mathbb{E}_v^\vee \otimes \mathbf{L}_{-1})}{-\mathbf{t}} \right)^5 \cdot \frac{5\mathbf{t}}{e_T(\mathbb{E}_v \otimes \mathbf{L}_5)} \cdot \left(\frac{1}{5\mathbf{t}} \right)^{|E_v|} \left(\frac{-\mathbf{t}^4}{5} \right)^{|S_v^{(1,\varphi)}|}, & v \in V_1^S; \\ \frac{1}{e_T(R\pi_{v*} \mathcal{L}_v^\vee \otimes \mathbf{L}_{-1})}, & v \in V_\infty^S. \end{cases}$$

Proof. We claim

$$(4.7) \quad \begin{aligned} & (H^i(\mathcal{V}|_{\mathcal{C}_v}) - H^i(\mathcal{O}_{\mathcal{C}_v}^{\oplus 2}))^{\text{mv}} \\ &= \begin{cases} H^i(\mathcal{L}|_{\mathcal{C}_v}) \otimes \mathbf{L}_1 = H^i(\mathcal{C}_v, f_v^* \mathcal{O}_{\mathbb{P}^4}(1)) \otimes \mathbf{L}_1, & v \in V_0^S; \\ (H^i(\mathcal{O}_{\mathcal{C}}(-\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_v}) \otimes \mathbf{L}_{-1})^{\oplus 5} \oplus H^i(\omega_{\mathcal{C}}(\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_v}) \otimes \mathbf{L}_5, & v \in V_1^S; \\ H^i(\mathcal{L}^\vee|_{\mathcal{C}_v}) \otimes \mathbf{L}_{-1}, & v \in V_\infty^S. \end{cases} \end{aligned}$$

We now derive these formulas.

1. In case $v \in V_0^S$, $\Sigma^{\mathcal{C}_v} \subset \Sigma_{(1,\rho)}^{\mathcal{C}}$ and $(\nu_1, \nu_2)|_{\mathcal{C}_v} = (0, 1)$. So $\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_v} \cong \mathcal{L}|_{\mathcal{C}_v}$, $\mathcal{N}|_{\mathcal{C}_v} \cong \mathcal{O}_{\mathcal{C}_v}$, and $\omega_{\mathcal{C}}^{\log}(-\Sigma_{(1,\rho)}^{\mathcal{C}})|_{\mathcal{C}_v} = \omega_{\mathcal{C}}|_{\mathcal{C}_v}$. Thus,

$$H^i(\mathcal{V}|_{\mathcal{C}_v}) = H^i(\mathcal{L}|_{\mathcal{C}_v})^{\oplus 5} \oplus H^i(\mathcal{L}^{\vee \otimes 5} \otimes \omega_{\mathcal{C}}|_{\mathcal{C}_v}) \oplus H^i(\mathcal{L}|_{\mathcal{C}_v}) \otimes \mathbf{L}_1 \oplus H^i(\mathcal{O}_{\mathcal{C}_v}).$$

Taking the moving parts, we obtain the formula (4.7) in case $v \in V_0^S$, which implies the formula of A'_v in case $v \in V_0^S$.

2. In case $v \in V_1^S$, $\Sigma^{\mathcal{C}_v} \subset \Sigma_{(1,\rho)}^{\mathcal{C}} \cup \Sigma_{(1,\varphi)}^{\mathcal{C}}$ and $(\nu_1, \nu_2)|_{\mathcal{C}_v} = (1, 1)$. So $(\mathcal{L} \otimes \mathcal{N})|_{\mathcal{C}_v} \otimes \mathbf{L}_1 \cong \mathcal{O}_{\mathcal{C}_v}$ and $\mathcal{N}|_{\mathcal{C}_v} \cong \mathcal{O}_{\mathcal{C}_v}$, which implies $\mathcal{L}|_{\mathcal{C}_v} \cong \mathcal{O}_{\mathcal{C}_v} \otimes \mathbf{L}_{-1}$. We have $S_v = S_v^{(1,\rho)} \cup S_v^{(1,\varphi)}$ where $S_v^{(1,\rho)}$ and $S_v^{(1,\varphi)}$ consist of markings in $\Sigma_{(1,\rho)}^{\mathcal{C}_v} := \mathcal{C}_v \cap \Sigma_{(1,\rho)}^{\mathcal{C}}$ and $\Sigma_{(1,\varphi)}^{\mathcal{C}_v} := \mathcal{C}_v \cap \Sigma_{(1,\varphi)}^{\mathcal{C}}$, respectively. Adding $\omega_{\mathcal{C}}^{\log}(-\Sigma_{(1,\rho)}^{\mathcal{C}})|_{\mathcal{C}_v} = \omega_{\mathcal{C}}(\Sigma_{(1,\rho)}^{\mathcal{C}})|_{\mathcal{C}_v}$, we obtain

$$H^i(\mathcal{V}|_{\mathcal{C}_v}) = (H^i(\mathcal{O}_{\mathcal{C}_v}(-\Sigma_{(1,\varphi)}^{\mathcal{C}_v})) \otimes \mathbf{L}_{-1})^{\oplus 5} \oplus H^i(\omega_{\mathcal{C}}(\Sigma_{(1,\varphi)}^{\mathcal{C}_v})|_{\mathcal{C}_v}) \otimes \mathbf{L}_5 \oplus H^i(\mathcal{O}_{\mathcal{C}_v})^{\oplus 2}.$$

Taking the moving parts, we obtain the formula (4.7) in case $v \in V_1^S$. To obtain the formula of A'_v in case $v \in V_1^S$, we need further simplification.

We have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{C}_v}(-\Sigma_{(1,\varphi)}^{\mathcal{C}_v}) \longrightarrow \mathcal{O}_{\mathcal{C}_v} \longrightarrow \mathcal{O}_{\Sigma_{(1,\varphi)}^{\mathcal{C}_v}} \longrightarrow 0.$$

Taking its induced long exact sequence, we get

$$(4.8) \quad \begin{aligned} & (H^0(\mathcal{O}_{\mathcal{C}_v}(-\Sigma_{(1,\varphi)}^{\mathcal{C}_v})) - H^1(\mathcal{O}_{\mathcal{C}_v}(-\Sigma_{(1,\varphi)}^{\mathcal{C}_v}))) \otimes \mathbf{L}_{-1} \\ &= -H^1(\mathcal{O}_{\mathcal{C}_v}) \otimes \mathbf{L}_{-1} - \mathbf{L}_{-1}^{\oplus (|S_v^{(1,\varphi)}| - 1)}. \end{aligned}$$

Because

$$\omega_{\mathcal{C}}(\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_v} = \omega_{\mathcal{C}_v}(\Sigma_{(1,\varphi)}^{\mathcal{C}_v}) + \sum_{e \in E_v} y_{(e,v)},$$

we further have the short exact sequence

$$0 \longrightarrow \omega_{\mathcal{C}_v} \longrightarrow \omega_{\mathcal{C}}(\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_v} \longrightarrow \mathcal{O}_{\Sigma_{(1,\varphi)}^{\mathcal{C}_v}} \oplus \bigoplus_{e \in E_v} \mathcal{O}_{y_{(e,v)}} \longrightarrow 0.$$

Taking its induced long exact sequence and using $H^1(\omega_{\mathcal{C}_v}) = \mathbb{C}$, we get

$$(4.9) \quad \begin{aligned} & (H^0(\omega_{\mathcal{C}}(\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_v}) - H^1(\omega_{\mathcal{C}}(\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_v})) \otimes \mathbf{L}_5 \\ &= H^0(\omega_{\mathcal{C}_v}) \otimes \mathbf{L}_5 \oplus \mathbf{L}_5^{\oplus (|S_v^{(1,\varphi)}| + |E_v| - 1)}. \end{aligned}$$

Combining (4.7) (in case $v \in V_1^S$), (4.8), and (4.9), we obtain the formula of A'_v in case $v \in V_1^S$.

3. In case $v \in V_\infty^S$, $\Sigma^{\mathbb{C}_v} \subset \Sigma_{\zeta_5}^{\mathbb{C}} \cap \Sigma_{\zeta_2}^{\mathbb{C}}$ since Γ is regular and $(\rho, \nu_1)|_{\mathbb{C}_v} = (1, 1)$, so $\mathcal{L}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_v} = \mathcal{L}|_{\mathbb{C}_v}$, $\mathcal{P}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_v} = \mathcal{P}|_{\mathbb{C}_v} \cong \mathcal{O}_{\mathbb{C}_v}$, and $(\mathcal{L} \otimes \mathcal{N})|_{\mathbb{C}_v} \otimes \mathbf{L}_1 \cong \mathcal{O}_{\mathbb{C}_v}$, equivalent to $\mathcal{N}|_{\mathbb{C}_v} \cong \mathcal{L}^\vee|_{\mathbb{C}_v} \otimes \mathbf{L}_{-1}$. Thus

$$H^i(\mathcal{V}|_{\mathbb{C}_v}) = H^i(\mathcal{L}|_{\mathbb{C}_v})^{\oplus 5} \oplus H^i(\mathcal{O}_{\mathbb{C}_v}) \oplus H^i(\mathcal{O}_{\mathbb{C}_v}) \oplus H^i(\mathcal{L}^\vee|_{\mathbb{C}_v}) \otimes \mathbf{L}_{-1}.$$

Taking the moving part, we get the formula (4.7) in case $v \in V_\infty^S$, which implies the formula of A'_v in case $v \in V_\infty^S$. \square

4.2.2. Contribution from edges. In this subsubsection, we compute the contribution A'_e from an edge $e \in E(\Gamma)$. We use

$$H^i(\mathcal{V}|_{\mathbb{C}_e}) = H^i(\mathcal{L}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_e})^{\oplus 5} \oplus H^i(\mathcal{P}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) \oplus H^i(\mathcal{N} \otimes \mathcal{L}|_{\mathbb{C}_e}) \otimes \mathbf{L}_1 \oplus H^i(\mathcal{N}|_{\mathbb{C}_e}).$$

Let v, v' and δ, δ' be defined as in Section 4.1, and $q = y_{(e, v)}$ and $q' = y_{(e, v')}$. We introduce some definitions.

(4.10)

$$\delta'_\varphi = \begin{cases} -1, & v' \in V_1^{1,1} \text{ and } q' \in \Sigma_{(1, \varphi)}^{\mathbb{C}}, \\ 0, & \text{otherwise;} \end{cases} \quad \delta_\varphi = \begin{cases} -1, & v \in V_\infty^{1,1} \text{ and } q \in \Sigma_{(1, \varphi)}^{\mathbb{C}}, \\ 0, & \text{otherwise.} \end{cases}$$

(4.11)

$$\delta'_\rho = \begin{cases} -1, & v' \in V_1^{1,1} \text{ and } q' \in \Sigma_{(1, \rho)}^{\mathbb{C}}, \\ 0, & \text{otherwise;} \end{cases} \quad \delta_\rho = \begin{cases} -1, & v \in V_0^{1,1} \text{ and } q \in \Sigma_{(1, \rho)}^{\mathbb{C}}, \\ 0, & \text{otherwise.} \end{cases}$$

With the above definition,

$$\begin{aligned} \mathcal{O}_{\mathbb{C}}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_e} &= \mathcal{O}_{\mathbb{C}_e}(\delta_\varphi q + \delta'_\varphi q'), \\ \mathcal{O}_{\mathbb{C}}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_e} &= \mathcal{O}_{\mathbb{C}_e}(\delta_\rho q + \delta'_\rho q'), \\ \omega_{\mathbb{C}}^{\log}|_{\mathbb{C}_e} &= \mathcal{O}_{\mathbb{C}_e}(\delta q + \delta' q'). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_e} &= \mathcal{L}|_{\mathbb{C}_e} \otimes \mathcal{O}_{\mathbb{C}_e}(\delta_\varphi q + \delta'_\varphi q'), \\ \mathcal{P}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_e} &= (\mathcal{L}^{\vee \otimes 5})|_{\mathbb{C}_e} \otimes \mathcal{O}_{\mathbb{C}_e}((\delta + \delta_\rho)q + (\delta' + \delta'_\rho)q') \end{aligned}$$

Lemma 4.3 (Contribution from an edge in E_0). *Given an edge $e \in E_0$, let $v \in V_0$ and $v' \in V_1$ be the two ends of e . Let δ' , δ'_φ , and δ'_ρ be defined as in (4.3), (4.10), and (4.11), respectively. Then*

$$A'_e = \frac{\prod_{j=1}^{5d_e-1-\delta'-\delta'_\rho} (-5h_e + \frac{j(h_e+t)}{d_e})}{\prod_{j=1}^{d_e+\delta'_\varphi} (h_e - \frac{j(h_e+t)}{d_e})^5 \cdot \prod_{j=1}^{d_e} \frac{j(h_e+t)}{d_e}}.$$

Note that δ' and δ'_φ can be both zero and cannot be both nonzero.

Proof. When $e \in E_0$, $(\rho, \nu_2)|_{\mathbb{C}_e} = (0, 1)$, so $\mathcal{N}|_{\mathbb{C}_e} \cong \mathcal{O}_{\mathbb{C}_e}$. Let $T^{1/d_e} = \mathbb{C}^* \rightarrow T = \mathbb{C}^*$ be defined by $\tilde{t} \mapsto \tilde{t}^{d_e}$. Then T^{1/d_e} acts on $\mathbb{C}_e = \mathbb{P}^1$ by $\tilde{t} \cdot [x, y] = [\tilde{t}x, y]$, and $H^2(B(T^{1/d_e}); \mathbb{Z}) = \mathbb{Z}(\mathfrak{t}/d_e)$. Let $q = [0, 1]$ and $q' = [1, 0]$ be the two T^{1/d_e} -fixed points on \mathbb{C}_e . Any T^{1/d_e} -linearized line bundle on $\mathbb{C}_e \cong \mathbb{P}^1$ is of the form $\mathcal{O}_{\mathbb{C}_e}(aq + bq')$ for some $a, b \in \mathbb{Z}$, characterized by

$$e_{T^{1/d_e}}(\mathcal{O}_{\mathbb{C}_e}(aq + bq')|_q) = \frac{at}{d_e}, \quad e_{T^{1/d_e}}(\mathcal{O}_{\mathbb{C}_e}(aq + bq')|_{q'}) = \frac{-bt}{d_e}.$$

We have the following isomorphisms of T^{1/d_e} -linearized line bundles on \mathbb{C}_e :

$$\begin{aligned} T\mathbb{C}_e &\cong \mathcal{O}_{\mathbb{C}_e}(q + q'), \quad \mathcal{L}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_e} \cong \mathcal{O}_{\mathbb{C}_e}((d_e + \delta'_\varphi)q'), \\ \mathcal{P}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_e} &\cong \mathcal{O}_{\mathbb{C}_e}((\delta + \delta_\rho)q + (-5d_e + \delta' + \delta'_\rho)q'). \end{aligned}$$

Recall that $\pi_e : \mathcal{W}_e \rightarrow \mathbb{P}^4$ is the projection to the coarse moduli space, and $\mathcal{E}_e \rightarrow \mathcal{W}_e$ is the tautological line bundle such that $\mathcal{E}_e^{\otimes d_e} = \pi_e^* \mathcal{O}_{\mathbb{P}^4}(1)$. By abuse of notations, we view $H^i(\mathcal{L}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_e})$ and $H^i(\mathcal{P}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_e})$ as T -equivariant vector bundles over \mathcal{W}_e . Then

$$H^0(\mathcal{L}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_e}) = \bigoplus_{j=0}^{d_e + \delta'_\varphi} \mathcal{E}_e^{\otimes(d_e - j)} \otimes (\mathbf{L}_{-1/d_e})^{\otimes j}, \quad H^1(\mathcal{L}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_e}) = 0;$$

$$H^0(\mathcal{P}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) = 0, \quad H^1(\mathcal{P}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) = \bigoplus_{j=1+\delta+\delta_\rho}^{5d_e-1-\delta'-\delta'_\rho} \mathcal{E}_e^{\otimes(-5d_e+j)} \otimes \mathbf{L}_{j/d_e},$$

and $H^0(\mathcal{O}_{\mathbb{C}_e})^{\text{mv}} = H^1(\mathcal{O}_{\mathbb{C}_e})^{\text{mv}} = 0$. So for $e \in E_0$,

$$(4.12) \quad (H^0(\mathcal{L}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_e}) - H^1(\mathcal{L}(-\Sigma_{(1, \varphi)}^{\mathbb{C}})|_{\mathbb{C}_e}))^{\text{mv}} = \bigoplus_{j=1}^{d_e + \delta'_\varphi} \mathcal{E}_e^{\otimes(d_e - j)} \otimes \mathbf{L}_{-j/d_e};$$

$$(4.13) \quad (H^0(\mathcal{N} \otimes \mathcal{L}|_{\mathbb{C}_e}) \otimes \mathbf{L}_1 - H^1(\mathcal{N} \otimes \mathcal{L}|_{\mathbb{C}_e}) \otimes \mathbf{L}_1)^{\text{mv}} = \bigoplus_{j=1}^{d_e} \mathcal{E}_e^{\otimes j} \otimes \mathbf{L}_{j/d_e};$$

(4.14)

$$(H^0(\mathcal{P}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) - H^1(\mathcal{P}(-\Sigma_{(1, \rho)}^{\mathbb{C}})|_{\mathbb{C}_e}))^{\text{mv}} = - \bigoplus_{j=1}^{5d_e-1-\delta'-\delta'_\rho} \mathcal{E}_e^{\otimes(-5d_e+j)} \otimes \mathbf{L}_{j/d_e}.$$

The Lemma follows from (4.12), (4.13), and (4.14). \square

Lemma 4.4 (Contribution from an edge in E_∞). *Given an edge $e \in E_\infty$, let $v \in V_\infty$ and $v' \in V_1$ be the two ends of e . Let δ' , δ'_φ , and δ'_ρ be defined as in*

(4.3), (4.10), and (4.11), respectively. Then

$$A'_e = \begin{cases} \frac{\prod_{j=1+\delta'_\varphi}^{\lceil -d_e \rceil - 1} (-t - \frac{j}{d_e})^5}{\prod_{j=1}^{-5d_e + \delta' + \delta'_\rho} (-\frac{j}{d_e}) \prod_{j=1}^{\lfloor -d_e \rfloor} (\frac{j}{d_e})}, & v \in V \setminus V^{0,1}; \\ \frac{\prod_{j=1+\delta'_\varphi}^{-d_e - 1} (-t - \frac{5j}{5d_e + 1})^5}{\prod_{j=1}^{-5d_e - 1 + \delta' + \delta'_\rho} (\frac{-5j}{5d_e + 1}) \prod_{j=1}^{-d_e} (\frac{5j}{5d_e + 1})}, & v \in V^{0,1}. \end{cases}$$

Note that δ' and δ'_ρ can be both zero and cannot be both nonzero.

Proof. Since $\nu_1|_{\mathcal{C}_e} = 1$, $(\mathcal{L} \otimes \mathcal{N} \otimes \mathbf{L}_1)|_{\mathcal{C}_e} \cong \mathcal{O}_{\mathcal{C}_e}$, and thus $\mathcal{L}|_{\mathcal{C}_e} = \mathcal{N}_{\mathcal{C}_e}^\vee \otimes \mathbf{L}_{-1}$, and

$$\begin{aligned} & H^i(\mathcal{V}|_{\mathcal{C}_e}) - H^i(\mathcal{O}_{\mathcal{C}_e}^{\oplus 2}) \\ &= H^i(\mathcal{N}^\vee(-\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_e} \otimes \mathbf{L}_{-1})^{\oplus 5} \oplus H^i(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathcal{C}})|_{\mathcal{C}_e}) \oplus H^i(\mathcal{N}|_{\mathcal{C}_e}) - H^i(\mathcal{O}_{\mathcal{C}_e}). \end{aligned}$$

Let $r_e \in \{1, 5\}$ be defined as in (4.4).

Case 1: $v \in V \setminus V^{0,1}$.

From Section 4.1,

$$\begin{aligned} e_T(T_{q'}\mathcal{C}_e) &= -\frac{t}{d_e}, \quad e_T(\mathcal{L}|_{q'}) = -t, \quad e_T(\mathcal{N}|_{q'}) = 0, \quad e_T(\mathcal{P}|_{q'}) = (5 - \frac{\delta'}{d_e})t; \\ e_T(T_q\mathcal{C}_e) &= \frac{t}{r_e d_e}, \quad e_T(\mathcal{L}|_q) = 0, \quad e_T(\mathcal{N}|_q) = -t, \quad e_T(\mathcal{P}|_q) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} e_T(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{q'}) &= (-1 - \frac{\delta'_\varphi}{d_e})t, \quad e_T(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathcal{C}})|_q) = \frac{\delta_\varphi t}{r_e d_e}; \\ e_T(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathcal{C}})|_{q'}) &= (5 - \frac{\delta' + \delta'_\rho}{d_e})t, \quad e_T(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathcal{C}})|_q) = 0. \end{aligned}$$

We have

$$H^0(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_e}) = 0, \quad H^1(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_e}) = \bigoplus_{j=1+\delta'_\varphi}^{\lceil -d_e \rceil - 1 - \delta_\varphi} \mathbf{L}_{-(1+j/d_e)},$$

so

$$(4.15) \quad (H^0(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_e}) - H^1(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathcal{C}})|_{\mathcal{C}_e}))^{\text{mv}} = - \bigoplus_{j=1+\delta'_\varphi}^{\lceil -d_e \rceil - 1} \mathbf{L}_{-(1+j/d_e)}.$$

We also have

$$(4.16) \quad H^0(\mathcal{N}|_{\mathcal{C}_e}) - H^0(\mathcal{O}_{\mathcal{C}_e}) = \bigoplus_{j=1}^{\lfloor -d_e \rfloor} \mathbf{L}_{j/d_e}, \quad H^1(\mathcal{N}|_{\mathcal{C}_e}) - H^1(\mathcal{O}_{\mathcal{C}_e}) = 0$$

Finally, $H^0(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) = \bigoplus_{j=5d_e-\delta'-\delta'_\rho}^0 \mathbf{L}_{j/d_e}$ and $H^1(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) = 0$. So

$$(4.17) \quad (H^0(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) - H^1(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_{\mathbb{C}_e}))^{\text{mv}} = \bigoplus_{j=5d_e-\delta'-\delta'_\rho}^{-1} \mathbf{L}_{j/d_e} = \bigoplus_{j=1}^{-5d_e+\delta'+\delta'_\rho} \mathbf{L}_{-j/d_e}.$$

Combining (4.15), (4.16), and (4.17), we obtain the Lemma in case $v \in V \setminus V^{0,1}$.

Case 2: $v \in V^{0,1}$.

In this case, $\delta = -1$, $\delta_\varphi = 0$, $-d_e \in \mathbb{Z}_{>0}$, $r_e = 1$. From Section 4.1,

$$\begin{aligned} e_T(T_{q'}\mathbb{C}_e) &= \frac{-5\mathbf{t}}{5d_e+1}, & e_T(\mathcal{L}|_{q'}) &= -\mathbf{t}, & e_T(\mathcal{N}|_{q'}) &= 0, & e_T(\mathcal{P}|_{q'}) &= (5 - \frac{5\delta'}{5d_e+1})\mathbf{t}; \\ e_T(T_q\mathbb{C}_e) &= \frac{5\mathbf{t}}{5d_e+1}, & e_T(\mathcal{L}|_q) &= \frac{-\mathbf{t}}{5d_e+1}, & e_T(\mathcal{N}|_q) &= \frac{-5d_e\mathbf{t}}{5d_e+1}, & e_T(\mathcal{P}|_q) &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} e_T(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathbb{C}})|_{q'}) &= -(1 + \frac{5\delta'_\varphi}{5d_e+1})\mathbf{t}, & e_T(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathbb{C}})|_q) &= e_T(\mathcal{L}|_q) = \frac{-\mathbf{t}}{5d_e+1}; \\ e_T(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_{q'}) &= (5 - \frac{5(\delta'+\delta'_\rho)}{5d_e+1})\mathbf{t}, & e_T(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_q) &= e_T(\mathcal{P}|_q) = 0. \end{aligned}$$

We have

$$H^0(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathbb{C}})|_{\mathbb{C}_e}) = 0, \quad H^1(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathbb{C}})|_{\mathbb{C}_e}) = \bigoplus_{j=1+\delta'_\varphi}^{-d_e-1} \mathbf{L}_{-(1+\frac{5j}{5d_e+1})},$$

so

$$(4.18) \quad (H^0(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathbb{C}})|_{\mathbb{C}_e}) - H^1(\mathcal{L}(-\Sigma_{(1,\varphi)}^{\mathbb{C}})|_{\mathbb{C}_e}))^{\text{mv}} = - \bigoplus_{j=1+\delta'_\varphi}^{-d_e-1} \mathbf{L}_{-(1+\frac{5j}{5d_e+1})}.$$

We also have

$$(4.19) \quad H^0(\mathcal{N}|_{\mathbb{C}_e}) - H^0(\mathcal{O}_{\mathbb{C}_e}) = \bigoplus_{j=1}^{-d_e} \mathbf{L}_{\frac{5j}{5d_e+1}}, \quad H^1(\mathcal{N}|_{\mathbb{C}_e}) = H^1(\mathcal{O}_{\mathbb{C}_e}) = 0.$$

Finally, $H^0(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) = \bigoplus_{j=5d_e+1-\delta'-\delta'_\rho}^0 \mathbf{L}_{\frac{5j}{5d_e+1}}$ and $H^1(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) = 0$. So

$$(4.20) \quad \begin{aligned} & (H^0(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_{\mathbb{C}_e}) - H^1(\mathcal{P}(-\Sigma_{(1,\rho)}^{\mathbb{C}})|_{\mathbb{C}_e}))^{\text{mv}} \\ &= \bigoplus_{j=5d_e+1-\delta'-\delta'_\rho}^{-1} \mathbf{L}_{\frac{5j}{5d_e+1}} = \bigoplus_{j=1}^{-5d_e-1+\delta'+\delta'_\rho} \mathbf{L}_{\frac{-5j}{5d_e+1}}. \end{aligned}$$

Combining (4.18), (4.19) and (4.20), we obtain the Lemma in case $v \in V^{0,1}$. \square

For later convenience, we rewrite the total contribution of all edges as follows.

Lemma 4.5 (Contribution from all edges).

$$\prod_{e \in E} A'_e = \prod_{e \in E} A_e \prod_{v \in V_1^{0,1} \cup V_1^{1,1}} A_v.$$

where A_e and A_v are defined as follows.

$$A_e = \begin{cases} \frac{\prod_{j=1}^{5d_e-1} (-5h_e + \frac{j(h_e+t)}{d_e})}{\prod_{j=1}^{d_e} (h_e - \frac{j(h_e+t)}{d_e})^5 \cdot \prod_{j=1}^{d_e} \frac{j(h_e+t)}{d_e}}, & e \in E_0; \\ \frac{\prod_{j=1}^{\lceil -d_e \rceil - 1} (-t - \frac{j t}{d_e})^5}{\prod_{j=1}^{-5d_e} (-\frac{j t}{d_e}) \prod_{j=1}^{\lfloor -d_e \rfloor} (\frac{j t}{d_e})}, & e \in E_\infty, (e, v) \in F, v \in V_\infty \setminus V_\infty^{0,1}; \\ \frac{\prod_{j=1}^{-d_e-1} (-t - \frac{5j t}{5d_e+1})^5}{\prod_{j=1}^{-5d_e-1} (-\frac{5j t}{5d_e+1}) \prod_{j=1}^{-d_e} (\frac{5j t}{5d_e+1})}, & e \in E_\infty, (e, v) \in F, v \in V_\infty^{0,1}. \end{cases}$$

$$A_v = \begin{cases} 5t, & v \in V_1^{0,1}; \\ -t^5, & v \in V_1^{1,1}, S_v \subset \Sigma_{(1,\varphi)}^c; \\ 5t, & v \in V_1^{1,1}, S_v \subset \Sigma_{(1,\rho)}^c. \end{cases}$$

Proof. This follows from Lemma 4.3, Lemma 4.4, and the definitions of δ' , δ'_φ , and δ'_ρ . \square

4.2.3. Contributions from nodes.

Lemma 4.6 (Contribution from a flag in F^S). *If $(e, v) \in F^S$ then*

$$A_{(e,v)} = \begin{cases} h_e + t, & v \in V_0; \\ -5t^6, & v \in V_1; \\ 1, & v \in V_\infty. \end{cases}$$

Proof. Let $y_{(e,v)} = \mathcal{C}_e \cap \mathcal{C}_v$ be the node associated to the flag $(e, v) \in F^S$. It is a scheme point when $v \in V_0 \cup V_1$; when $v \in V_\infty$, it is a stacky point $B\mu_5$ because $d_e \notin \mathbb{Z}$ as Γ is regular. Note that $H^0(\mathcal{V}|_{y_{(e,v)}}) = H^0(\mathcal{O}_{y_{(e,v)}}^{\oplus 2})$ when $y_{(e,v)} = B\mu_5$.

$$(4.21) \quad (H^0(\mathcal{V}|_{y_{(e,v)}}) - H^0(\mathcal{O}_{y_{(e,v)}}^{\oplus 2}))^{\text{mv}} = \begin{cases} \text{ev}_{(e,v)}^* \mathcal{O}_{\mathbb{P}^4}(1) \otimes \mathbf{L}_1, & v \in V_0; \\ \mathbf{L}_{-1}^{\oplus 5} \oplus \mathbf{L}_5, & (e, v) \in V_1; \\ 0, & (e, v) \in V_\infty. \end{cases}$$

Here, when $v \in V_0$, \mathcal{W}_v is the moduli of stable maps to \mathbb{P}^4 with p -fields, and $ev_{(e,v)}$ is the evaluation map at the marking labeled by e . The Lemma follows from (4.21). \square

Lemma 4.7 (Contribution from a vertex in $V^{0,2}$). *If $v \in V^{0,2}$, then*

$$A_v = \begin{cases} h_e + \mathbf{t} = h_{e'} + \mathbf{t}, & v \in V_0^{0,2} \text{ and } E_v = \{e, e'\}, \\ -5\mathbf{t}^6, & v \in V_1^{0,2}, \\ (-\mathbf{t})^{6\epsilon(d_e)}, & v \in V_\infty^{0,2} \text{ and } E_v = \{e, e'\}. \end{cases}$$

Here we define $\epsilon(x) = 1$ when $x \in \mathbb{Z}$, and $\epsilon(x) = 0$ otherwise.

Proof. If $v \in V^{0,2}(\Gamma)$ and $E_v = \{e, e'\}$, then $\mathcal{C}_v = \mathcal{C}_e \cap \mathcal{C}_{e'}$ is a node, which is a stacky point $B\mu_5$ if and only if $v \in V_\infty^{0,2}(\Gamma)$ and $d_e \neq \mathbb{Z}$. (The balance condition implies $d_e - d_{e'} \in \mathbb{Z}$.) We have

$$(H^0(\mathcal{V}|_{\mathcal{C}_v}) - H^0(\mathcal{O}_{\mathcal{C}_v}^{\oplus 2}))^{\text{mv}} = \begin{cases} \mathcal{O}_{\mathbb{P}^4}(1) \otimes \mathbf{L}_1, & v \in V_0^{0,2} \text{ (so that } \mathcal{W}_v = \mathbb{P}^4); \\ \mathbf{L}_{-1}^{\oplus 5} \oplus \mathbf{L}_5, & v \in V_1^{0,2}; \\ 0, & v \in V_\infty^{0,2}, d_e \notin \mathbb{Z}; \\ \mathbf{L}_{-1}^{\oplus 6}, & v \in V_\infty^{0,2}, d_e \in \mathbb{Z}. \end{cases}$$

This proves the Lemma. \square

4.2.4. *Summary.* For $v \in V^S$, let A'_v be as in Lemma 4.2, and define (4.22)

$$A_v := A'_v \prod_{e \in E_v} A_{(e,v)} \\ = \begin{cases} \frac{1}{e_T(R\pi_{v*}\phi_v^*\mathcal{O}_{\mathbb{P}^4}(1) \otimes \mathbf{L}_1)} \prod_{e \in E_v} (h_e + \mathbf{t}), & v \in V_0^S; \\ \left(\frac{e_T(\mathbb{E}_v^\vee \otimes \mathbf{L}_{-1})}{-\mathbf{t}} \right)^5 \cdot \frac{5\mathbf{t}}{e_T(\mathbb{E}_v \otimes \mathbf{L}_5)} \cdot (-\mathbf{t}^5)^{|E_v|} \cdot \left(\frac{-\mathbf{t}^4}{5} \right)^{|S_v^{(1,\varphi)}|}, & v \in V_1^S; \\ \frac{1}{e_T(R\pi_{v*}\mathcal{L}_v^\vee \otimes \mathbf{L}_{-1})}, & v \in V_\infty^S. \end{cases}$$

Then

$$(4.23) \quad \prod_{v \in V^S} A_v = \prod_{v \in V^S} A'_v \prod_{(e,v) \in F^S} A_{(e,v)}.$$

Proposition 4.8. *We define A_v by (4.22) if $v \in V^S$. Let A_v be given by Lemma 4.5 (resp. Lemma 4.7) if $v \in V_1^{0,1} \cup V_1^{1,1}$ (resp. $v \in V^{0,2}$). If $v \in V_0^{0,1} \cup V_0^{1,1} \cup V_\infty^{0,1} \cup V_\infty^{1,1}$, we define $A_v = 1$. For $e \in E$, let A_e be as in Lemma 4.5. Then*

$$\frac{e_T(B_2^{\text{mv}})e_T(B_6^{\text{mv}})}{e_T(B_3^{\text{mv}})e_T(B_5^{\text{mv}})} = \prod_{v \in V} A_v \prod_{e \in E} A_e.$$

Proof. This follows from (4.6), Lemma 4.5, and (4.23). \square

4.3. Equivariant Euler class of the virtual normal bundle. Let A_v and A_e be defined as in Proposition 4.8. Combining (4.2) in Section 4.1 and Proposition 4.8, we obtain

Theorem 4.9.

$$\frac{1}{e_T(N_\Gamma^{\text{vir}})} = \prod_{v \in V(\Gamma)} B_v \prod_{e \in E(\Gamma)} A_e,$$

where

$$B_v = \begin{cases} A_v \cdot \prod_{e \in E_v} \frac{1}{w_{(e,v)} - \psi_{(e,v)}}, & v \in V^S; \\ \frac{A_v}{w_{(e,v)} + w_{(e,v')}}, & v \in V^{0,2}, E_v = \{e, e'\}; \\ A_v, & v \in V^{1,1}; \\ A_v \cdot w_{(e,v)}, & v \in V^{0,1}, E_v = \{e\}. \end{cases}$$

5. EFFECTIVE RELATIONS AMONG GW AND FJRW OF QUINTICS

We will show that for a regular graph $\Gamma \in \Delta^{\text{reg}}$, the contributions to $[\mathcal{W}_\Gamma]_{\text{loc}}^{\text{vir}}$ from vertices in $V_\infty^S(\Gamma)$ are twisted FJRW invariants, and those from vertices in $V_0^S(\Gamma)$ are GW invariants of the quintic. We then investigate the relations obtained from MSP field theory between GW invariants of the quintic threefold and FJRW invariants of $([\mathbb{C}^5/\mathbb{Z}_5], \mathfrak{w}_5)$.

5.1. FJRW invariants of $([\mathbb{C}^5/\mathbb{Z}_5], \mathfrak{w}_5)$. This class of FJRW invariants was constructed in [PV, Chi, Mo, FJR2]. We will follow its latter construction using the cosection localized virtual cycles [CLL].

Let $(\mathbb{G}_m)^5$ act on \mathbb{C}^5 by the standard product action, and let $G_5 \leq (\mathbb{G}_m)^5$ be generated by $(\zeta_5, \zeta_5, \zeta_5, \zeta_5, \zeta_5)$. Let $\gamma = (\gamma_1, \dots, \gamma_\ell) \in (\boldsymbol{\mu}_5^\times)^\ell$, where $\boldsymbol{\mu}_5^\times = \boldsymbol{\mu}_5 - \{1\}$. A γ -pointed genus- g G_5 -spin curve is a stable genus g ℓ -pointed twisted nodal curve with line bundles $\mathcal{L}_1, \dots, \mathcal{L}_5$ and isomorphisms

$$(5.1) \quad \phi_0 : \mathcal{L}_1^5 \cong \omega_{\mathcal{C}}^{\log} \quad \text{and} \quad \phi_i : \mathcal{L}_i^{-1} \otimes \mathcal{L}_{i+1} \cong \mathcal{O}_{\mathcal{C}}$$

such that the monodromy of \mathcal{L}_i along $\Sigma_i^{\mathcal{C}}$ is γ_i . The reason that the relations take this form is that the group of G_5 -invariant Laurent monomials in (x_1, \dots, x_5) is generated by x_1^5 and $x_i^{-1}x_{i+1}$'s. Note the relations (5.1) are equivalent to

$$\mathcal{L} := \mathcal{L}_1 \cong \dots \cong \mathcal{L}_5 \quad \text{and} \quad \mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}^{\log}.$$

We form the category whose objects are flat families of stable genus- g γ -pointed G_5 -spin curves, and arrows between them are arrows of the curves together with isomorphisms of the line bundles that commute with the ϕ_i 's. It is a smooth, proper DM stack, which we denote by $\overline{\mathcal{M}}_{g,\gamma}(G_5)$.

The (narrow) FJRW invariants of $([\mathbb{C}^5/\mathbb{Z}_5], \mathfrak{w}_5)$ is defined via the cosection localized virtual cycle

$$[\overline{\mathcal{M}}_{g,\gamma}(G_5)^p]_{\text{loc}}^{\text{vir}} \in A_* \overline{\mathcal{M}}_{g,\gamma}(G_5)$$

of the moduli $\overline{\mathcal{M}}_{g,\gamma}(G_5)^p$ of G_5 -spin curves with fields. By definition $\overline{\mathcal{M}}_{g,\gamma}(G_5)^p$ parameterizes families $(\mathcal{C}, \Sigma^{\mathcal{C}}, (\mathcal{L}_i, \varphi_i)_{i=1}^5)$, where $(\mathcal{C}, \Sigma^{\mathcal{C}}, (\mathcal{L}_i)_{i=1}^5)$ are families in $\overline{\mathcal{M}}_{g,\gamma}(G_5)$ and $\varphi_i \in H^0(\mathcal{C}, \mathcal{L}_i)$. It is called narrow because all $\gamma_i \in \mu_5^\times$.

Let $\overline{\mathcal{M}}_{g,\gamma}^{1/5}$ be the stack of families of $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L} : \mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}^{\log})$, of ℓ -marked genus- g twisted nodal curves together with a line bundle \mathcal{L} satisfying $\mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}^{\log}$ and having monodromy γ_i along the marking $\Sigma_i^{\mathcal{C}}$. Let $\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p}$ be the moduli of 5-spin twisted curves with five p -fields: i.e. its closed points are $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \varphi_1, \dots, \varphi_5)$ with $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}) \in \overline{\mathcal{M}}_{g,\gamma}^{1/5}$ and $\varphi_i \in H^0(\mathcal{L})$.

We have obvious morphisms $\tilde{\phi}_{g,\gamma}$ and $\phi_{g,\gamma}$:

$$\tilde{\phi}_{g,\gamma} : \overline{\mathcal{M}}_{g,\gamma}^{1/5,5p} \longrightarrow \overline{\mathcal{M}}_{g,\gamma}(G_5)^p \quad \text{and} \quad \phi_{g,\gamma} : \overline{\mathcal{M}}_{g,\gamma}^{1/5} \longrightarrow \overline{\mathcal{M}}_{g,\gamma}(G_5),$$

where the former sends $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \varphi_1, \dots, \varphi_5)$ to $(\mathcal{C}, \Sigma^{\mathcal{C}}, (\mathcal{L}, \varphi_1), \dots, (\mathcal{L}, \varphi_5))$.

Lemma 5.1. *Both $\phi_{g,\gamma}$ and $\tilde{\phi}_{g,\gamma}$ are isomorphisms of DM stacks. Furthermore,*

$$\phi_{g,\gamma}^*[\overline{\mathcal{M}}_{g,\gamma}(G_5)^p]_{\text{loc}}^{\text{vir}} = [\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p}]_{\text{loc}}^{\text{vir}} \in A_* \overline{\mathcal{M}}_{g,\gamma}^{1/5}.$$

Proof. The proof of the first part is a tautology. The proof of the second part is via comparing the perfect relative obstruction theories of the two stacks and the cosections of their respective obstruction sheaves. The verification is routine, and will be omitted. \square

We fix conventions that we will be using throughout this section. We write

$$(5.2) \quad \gamma = (\zeta_5^{m_1}, \dots, \zeta_5^{m_\ell}) \in (\mu_5^\times)^\ell, \quad m_i \in [1, 4].$$

We will also use the standard abbreviation $\gamma = (\zeta_5, \zeta_5, \zeta_5^2, \zeta_5^3) = (1123) = (1^223)$. We assume $2g + \ell > 3$, unless otherwise stated.

Proposition 5.2. *When $2g - 2 - \sum_{i=1}^\ell (m_i - 1) \not\equiv 0 \pmod{5}$, $\overline{\mathcal{M}}_{g,\gamma}(G_5)^p = \emptyset$. When $\overline{\mathcal{M}}_{g,\gamma}(G_5)^p$ is not empty, its virtual dimension is $\sum_{i=1}^\ell (2 - m_i)$.*

Proposition 5.3. *The cycle $[\overline{\mathcal{M}}_{g,\gamma}(G_5)^p]_{\text{loc}}^{\text{vir}} = 0$ unless (g, γ) is one of the following cases:*

- (1) $g \geq 1$ and $\gamma_i \in \{\zeta_5, \zeta_5^2\}$ for all i ;
- (2) $g = 0$ and $\gamma = (1^{1+k}23)$ or $(1^{2+k}4)$, $k \geq 0$.

These vanishings are known to experts. For instance see [CLL, Def. 2.7, Lem. 2.8, (3.3)].

Remark 5.4. *In case $g = 0$ and $\gamma = (123)$ or (1^24) , one easily sees that $\overline{\mathcal{M}}_{g,\gamma}^{1/5} \cong B\mu_5$. Thus $\deg[\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p}]_{\text{loc}}^{\text{vir}} = \frac{1}{5}$.*

5.2. Twisted FJRW invariants. Recall FJRW invariants with descendants. Given $\gamma \in (\mu_5^\times)^\ell$, and $2g - 2 + \ell > 0$, define

$$(5.3) \quad \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_\ell}(\gamma_\ell) \rangle_g^{G_5} := \int_{[\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p}]_{\text{loc}}^{\text{vir}}} \psi_1^{a_1} \cdots \psi_\ell^{a_\ell},$$

where ψ_i is the first Chern class of the relative cotangent bundle of the universal curve of $\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p} (= \overline{\mathcal{M}}_{g,\gamma}(G_5)^p)$ along its i -th marking. Later, we will use $\bar{\psi}$ to denote the psi class given by cotangent line bundle of the coarse marked point. In our case, (as all $\gamma_i \in \mu_5^\times$), $\bar{\psi}_i = 5\psi_i$.

Given $k \geq 0$ with $2g - 2 \equiv k(5)$ and $\gamma = (2^k)$, then $[\overline{\mathcal{M}}_{g,(2^k)}^{1/5,5p}]_{\text{loc}}^{\text{vir}}$ is zero dimensional. We define the primitive FJRW invariant to be

$$(5.4) \quad \Theta_{g,k} = \int_{[\overline{\mathcal{M}}_{g,(2^k)}^{1/5,5p}]_{\text{loc}}^{\text{vir}}} 1 \in \mathbb{Q}.$$

Note that all $\Theta_{0,k \geq 3}$ were calculated in [CR, Thm. 1.1.1]. For convenience, we set $\Theta_{1,0} = 1$.

Let $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L})$ with $\pi : \mathcal{C} \rightarrow \overline{\mathcal{M}}_{g,\gamma}^{1/5}$ be the universal family of $\overline{\mathcal{M}}_{g,\gamma}^{1/5}$. Set

$$(5.5) \quad \mathfrak{e}_T(\mathcal{L}^\vee) := e_{\mathbb{G}_m}(R\pi_* \mathcal{L}^\vee \otimes \mathbf{L}_{-1}) \in H_{\mathbb{G}_m}^*(\overline{\mathcal{M}}_{g,\gamma}^{1/5}).$$

We define the dual-twisted FJRW invariants to be

$$(5.6) \quad \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_\ell}(\gamma_\ell) \rangle_g^{\text{d.t., } G_5} := \int_{[\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p}]_{\text{loc}}^{\text{vir}}} \mathfrak{e}_T(\mathcal{L}^\vee)^{-1} \cdot \psi_1^{a_1} \cdots \psi_\ell^{a_\ell}.$$

By Proposition 3.5 and (4.22), they are the relevant contributions associate to $v \in V_0^S(\Gamma)$ in the localization formula (4.22). By Proposition 5.3, when $g \geq 1$, (5.3) and (5.6) vanish unless all $\gamma_i \in \{\zeta_5, \zeta_5^2\}$.

We now use Grothendieck-Riemann-Roch formula to express (5.6) in terms of $\Theta_{g,k}$. We first calculate the Chern character of $R\pi_*(\mathcal{L}^\vee)$ over $\overline{\mathcal{M}}_{g,\gamma}^{1/5}$. Following the notations introduced in [Chi, Theo 1.1.1], let Υ be the stack of objects $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, z, \mathcal{U}_z)$, where $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}) \in \overline{\mathcal{M}}_{g,\gamma}^{1/5}$, z is a node of \mathcal{C} , and \mathcal{U}_z is a branch of the formal completion $\hat{\mathcal{C}}_z$ of \mathcal{C} at z . Clearly, Υ is a smooth DM-stack.

Let $\psi_z \in A^1\Upsilon$ be the first Chern class (the ψ class) of the relative tangent bundle (i.e. $T_z\mathcal{U}_z$) of the distinguished branch along the distinguished node of the universal family of Υ . Let $\bar{\psi}_z$ to be the coarse moduli analogue of ψ_z . Namely, let $\bar{\mathcal{U}}_z$ be the coarse moduli of \mathcal{U}_z , then $\bar{\psi}_z$ is the first Chern class of the family of $T_z\bar{\mathcal{U}}_z$, the cotangent line bundle after taking the coarse moduli of the distinguished marking. (Thus $\bar{\psi}_z$ equals $5\psi_z$ when the node is non-scheme, and equals ψ_z otherwise.)

Let $j : \Upsilon \rightarrow \overline{\mathcal{M}}_{g,\gamma}^{1/5}$ be the forgetful morphism. The image $j(\Upsilon) \subset \overline{\mathcal{M}}_{g,\gamma}^{1/5}$ is the divisor of spin curves with singular domains, and $j : \Upsilon \rightarrow j(\Upsilon)$ is finite. Let $\Upsilon_l \subset \Upsilon$ be the open and close substack of objects $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, z, \mathcal{U}_z)$ so that

the monodromy of $\mathcal{L}|_{\mathcal{U}_z}$ along z is ζ_5^l , and let $\mathcal{I}_l = \mathcal{I}|_{\Upsilon_l} : \Upsilon_l \rightarrow \overline{\mathcal{M}}_{g,\gamma}^{1/5}$. Let $\sigma : \Upsilon \rightarrow \Upsilon$ be the involution that sends $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, z, \mathcal{U}_z)$ to $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}, z, \mathcal{U}'_z)$, where \mathcal{U}_z and \mathcal{U}'_z are the two branches of the distinguished node of \mathcal{C} .

For $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}) \in \overline{\mathcal{M}}_{g,\gamma}^{1/5}$, let $\bar{\mathcal{C}}$ be the twisted curve after taking the coarse moduli of \mathcal{C} at all its markings, and let $\Sigma_i^{\bar{\mathcal{C}}} \subset \bar{\mathcal{C}}$ be the associated i -th (scheme) marking. We abbreviate $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{g,\gamma}^{1/5}$. Then the collection of such cuves $(\Sigma_i^{\bar{\mathcal{C}}}, \bar{\mathcal{C}})$ form a family of curves $\bar{\pi} : \bar{\mathcal{C}} \rightarrow \overline{\mathcal{M}}$, with $\bar{\Sigma}_i \subset \bar{\mathcal{C}}$ being the associated sections of markings. Let

$$(5.7) \quad \bar{\psi}_i = c_1(T_{\bar{\mathcal{C}}/\overline{\mathcal{M}}}|_{\bar{\Sigma}_i}), \quad \kappa_h = \pi_*(c_1(\omega_{\bar{\mathcal{C}}/\overline{\mathcal{M}}}^{\log})^{h+1}) = \bar{\pi}_*(c_1(\omega_{\bar{\mathcal{C}}/\overline{\mathcal{M}}}^{\log})^{h+1}).$$

As in (5.2), $\gamma_i = \zeta_5^{m_i}$ where m_i are integers in $[1, 4]$. Let $B_m(x)$ be the Bernoulli polynomials.

Lemma 5.5. *The h -th total Chern class*

$$\begin{aligned} ch_h(R\pi_*\mathcal{L}^\vee) &= \frac{B_{h+1}(-1/5)}{(h+1)!} \kappa_h - \sum_{i=1}^{\ell} \frac{B_{h+1}((5-m_i)/5)}{(h+1)!} (\bar{\psi})_i^h + \\ &\quad + \frac{1}{2} \sum_{k=0}^4 \frac{5B_{h+1}(k/5)}{(h+1)!} 5^{-\delta_{k,0}} (J_k)_* \left(\sum_{i+j=h-1} (-\bar{\psi})^i \sigma^* (\bar{\psi})^j \right). \end{aligned}$$

Proof. We prove this lemma by applying the formula in [Chi, Theo 1.1.1]. We first recall its setuuf. Following [Chi, Definition 2.1.1], a 5-stable ℓ -pointed curve $(\mathcal{A}, x_1, \dots, x_\ell)$ is an ℓ -pointed twisted nodal curve with all of its markings being scheme points, all of its nodes being μ_5 balanced, and its automorphisms group being finite. Following [Chi, Theorem 2.2.1], we form the groupoid $\overline{\mathcal{N}}$ of $(\mathcal{A}, x_i, \mathcal{S})$, where (\mathcal{A}, x_i) is a 5-stable ℓ -pointed curve, and \mathcal{S} is an invertible sheave on \mathcal{A} so that $\mathcal{S}^{\otimes 5} \cong (\omega_{\mathcal{A}}^{\log})^{-1}(-\sum_{i=1}^{\ell} (5-m_i)x_i)$. The stack $\overline{\mathcal{N}}$ is a smooth proper DM stack. Let $\hat{\pi} : \mathcal{A} \rightarrow \overline{\mathcal{N}}$, $X_1, \dots, X_\ell \subset \mathcal{A}$ and \mathcal{S} be the universal family of $\overline{\mathcal{N}}$, which comes with the isomorphism

$$\mathcal{S}^{\otimes 5} \xrightarrow{\cong} (\omega_{\mathcal{A}/\overline{\mathcal{N}}}^{\log})^{-1} \left(- \sum_{i=1}^{\ell} (5-m_i)X_i \right).$$

Let $\eta : \mathcal{C} \rightarrow \bar{\mathcal{C}}$ be the tautological partial coarse moduli morphism, and let $\bar{\mathcal{S}} = \eta_*(\mathcal{L}^\vee)$ which is a line bundle on $\bar{\mathcal{C}}$. Using $\mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}/\overline{\mathcal{M}}}^{\log}$, we obtain

$$\bar{\mathcal{S}}^{\otimes r} \cong (\omega_{\bar{\mathcal{C}}/\overline{\mathcal{M}}}^{\log})^{-1} \left(- \sum_{i=1}^{\ell} (5-m_i)\bar{\Sigma}_i \right).$$

Thus the family $(\bar{\mathcal{C}}, \bar{\Sigma}_i, \bar{\mathcal{S}})$ would induce a morphism $\overline{\mathcal{N}} \rightarrow \overline{\mathcal{M}}$ should $\bar{\mathcal{S}}$ be representable along all nodes of the base curves.

To remedy this, to each $(\mathcal{A}, x_i, \mathcal{S}) \in \overline{\mathcal{N}}$, let $\hat{\mathcal{A}}$ be the twisted curve obtained from \mathcal{A} by forgetting the stacky structure at every node of \mathcal{A} where \mathcal{S} has trivial monodromy, and let $\zeta : \mathcal{A} \rightarrow \hat{\mathcal{A}}$ be the obvious morphism. Then $(\hat{\mathcal{A}}, x_i, \zeta_*\mathcal{S}) \cong (\bar{\mathcal{C}}, \Sigma_i^{\bar{\mathcal{C}}}, \bar{\mathcal{L}})$ for a unique point $(\mathcal{C}, \Sigma_i^{\mathcal{C}}, \mathcal{L}) \in \overline{\mathcal{M}}$. This association produces a

morphism $f : \overline{\mathcal{N}} \rightarrow \overline{\mathcal{M}}$, isomorphic on a dense open subset, so that

$$(5.8) \quad R\hat{\pi}_* \mathcal{S} = f^*(R\hat{\pi}_* \bar{\mathcal{S}}) = f^*(R\pi_* \mathcal{L}^\vee).$$

We continue to recall the notations in [Chi, page 3]. Like the case for $\Upsilon \rightarrow \overline{\mathcal{M}}$, let $\Upsilon' = \bigcup_{k=0}^4 \Upsilon'_k$ be the smooth DM stack of objects $(\mathcal{A}, x_i, \mathcal{S}, z, \mathcal{U}_z)$, where $(\mathcal{A}, x_i, \mathcal{S}) \in \overline{\mathcal{N}}$, and (z, \mathcal{U}_z) is the pair of a node and a branch of \mathcal{A} . Here $\Upsilon'_k \subset \Upsilon'$ is the substack characterized by the monodromy of $\mathcal{S}|_{\mathcal{U}_z}$ along z being ζ_5^k . Let $j : \Upsilon' \rightarrow \overline{\mathcal{N}}$ be the forgetful morphism.

Let $j_k = j|_{\Upsilon'_k}$, and let $\tilde{\sigma} : \Upsilon' \rightarrow \Upsilon'$ be the evolution.

By [Chi, Theo 1.1.1], the GRR formula takes the form

$$\begin{aligned} ch_h(R\hat{\pi}_* \mathcal{S}) &= \frac{B_{h+1}(-1/5)}{(h+1)!} \tilde{\kappa}_h - \sum_{i=1}^{\ell} \frac{B_{h+1}((5-m_i)/5)}{(h+1)!} (\tilde{\psi}_i)^h + \\ &\quad + \frac{1}{2} \sum_{k=0}^4 \frac{5B_{h+1}(k/5)}{(h+1)!} (j_k)_* \left(\sum_{i+j=h-1} (-\tilde{\psi}_z)^i \tilde{\sigma}^*(\tilde{\psi}_z)^j \right), \end{aligned}$$

where $\tilde{\psi}_i$ is the i -th ψ -class of $\overline{\mathcal{N}}$, and $\tilde{\psi}_z$ is the ψ -class of the distinguished node-branch of Υ' (cf. [Chi, page 3 line 20]). By our construction we have $\tilde{\psi}_i = f^* \bar{\psi}_i$, $\tilde{\psi}_z = \lambda_k^* \bar{\psi}_z$, where $\lambda_k : \Upsilon'_k \rightarrow \Upsilon_k$ is defined similar to that of $f : \overline{\mathcal{N}} \rightarrow \overline{\mathcal{M}}$. $\tilde{\kappa}_k$ is the standard κ -class of $\overline{\mathcal{N}}$.

Applying f_* to the above formula, one checks that the first two terms on the right hand side of the identity coincide with that in the startment of the lemma. For the third we use the commutative square

$$\begin{array}{ccc} \Upsilon'_k & \xrightarrow{j_k} & \overline{\mathcal{N}} \\ \downarrow \lambda_k & & \downarrow f \\ \Upsilon_k & \xrightarrow{j_k} & \overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,\gamma}^{1/5}. \end{array}$$

For $k \neq 0$, λ_k is birational, thus

$$f_* j_{k*} \left((-\tilde{\psi})^i \sigma^*(\tilde{\psi})^j \right) = j_{k*} \lambda_{k*} \left((-\tilde{\psi})^i \sigma^*(\tilde{\psi})^j \right) = j_{k*} \left((-\bar{\psi})^i \sigma^*(\bar{\psi})^j \right),$$

as desired.

In case $k = 0$, λ_0 is generically a μ_5 -gerbe (c.f. [Chi, Def 2.1.1], [ACV, Thm. 7.1.1]). Therefore

$$f_* j_{0*} \left((-\tilde{\psi})^i \sigma^*(\tilde{\psi})^j \right) = j_{0*} \lambda_{0*} \lambda_0^* \left((-\bar{\psi})^i \sigma^*(\bar{\psi})^j \right) = \frac{1}{5} j_{0*} \left((-\bar{\psi})^i \sigma^*(\bar{\psi})^j \right).$$

This gives the term $5^{-1} = 5^{-\delta_{0,0}}$. This completes the proof. \square

We now reduce dual-twisted FJRW invariants to primitive FJRW invariants. We need the following simple fact.

Lemma 5.6. *The moduli $\overline{\mathcal{M}}_{1,\zeta_5}^{1/5}$ is a disjoint union of two one dimensional smooth DM-stacks M_0 and M_1 , characterized by that $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}) \in M_0$ (resp.*

$\in M_1$) if $\mathcal{L}(-\Sigma^{\mathcal{C}}) \cong \mathcal{O}_{\mathcal{C}}$ (resp. $\mathcal{L}(-\Sigma^{\mathcal{C}})$ has no section). Furthermore,

$$(5.9) \quad [\overline{\mathcal{M}}_{1,\zeta_5}^{1/5,5p}]_{\text{loc}}^{\text{vir}} = -4^5[M_0] + [M_1].$$

Proof. By the assumption on the monodromy, $\mathcal{L}(-\Sigma^{\mathcal{C}})$ has trivial monodromy along $\Sigma^{\mathcal{C}}$, and $\mathcal{L}(-\Sigma^{\mathcal{C}})^{\otimes 5} \cong \mathcal{O}_{\mathcal{C}}$. Thus $\overline{\mathcal{M}}_{1,\zeta_5}^{1/5}$ is isomorphic to the moduli of $(\mathcal{C}, \Sigma, \mathcal{L}')$ so that (\mathcal{C}, Σ) is a one-pointed genus 1 twisted curve and $\mathcal{L}'^{\otimes 5} \cong \mathcal{O}_{\mathcal{C}}$. This gives $\overline{\mathcal{M}}_{1,\zeta_5}^{1/5} = M_0 \cup M_1$ (cf. [AJ]).

For (5.9), we first note that since $H^0(\mathcal{L}') = 0$ for $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}') \in M_1$,

$$[\overline{\mathcal{M}}_{1,\zeta_5}^{1/5,5p}]_{\text{loc}}^{\text{vir}} = c[M_0] + [M_1], \quad c \in \mathbb{Q}.$$

Since for $(\Sigma^{\mathcal{C}}, \mathcal{C}, \mathcal{L}') \in M_0$, $h^i(\mathcal{L}') = 1$ for $i = 0$ and 1 , [CLL, Prop 4.17] gives $c = -4^5$. \square

Lemma 5.7. *Given $a_1, \dots, a_\ell \in \mathbb{Z}_{\geq 0}$ and $\gamma_1, \dots, \gamma_\ell \in \{\zeta_5, \zeta_5^2\}$, letting $k = \#\{\gamma_i \mid \gamma_i = \zeta_5^2\}$, there is a $\mathbf{c} \in \mathbb{Q}[\mathbf{t}, \mathbf{t}^{-1}]$, depending on $\{g, \ell, k, a_1, \dots, a_k\}$ and effectively calculable, such that*

$$(5.10) \quad \langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_\ell}(\gamma_\ell) \rangle_g^{\text{d.t., } G_5} = \Theta_{g,k} \cdot \mathbf{c}.$$

Proof. Assume $(g, k) \neq (1, 0)$. From [CLL, Theo 4.5], we have the forgetful morphism

$$\hat{\pi} : \overline{\mathcal{M}}_{g,\gamma}^{1/5} \longrightarrow \overline{\mathcal{M}}_{g,(2^k)}^{1/5},$$

after forgetting all ζ_5 -markings in $(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}) \in \overline{\mathcal{M}}_{g,\gamma}^{1/5}$. By [CLL, Thm. 4.10], we have $\hat{\pi}^*[\overline{\mathcal{M}}_{g,(2^k)}^{1/5,5p}]_{\text{loc}}^{\text{vir}} = [\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p}]_{\text{loc}}^{\text{vir}}$. Because $\overline{\mathcal{M}}_{g,\gamma}^{1/5}$ is smooth, we have the commutative square

$$\begin{array}{ccc} A_*(\overline{\mathcal{M}}_{g,(2^k)}^{1/5}) & \xrightarrow{\hat{\pi}^*} & A_*(\overline{\mathcal{M}}_{g,\gamma}^{1/5}) \\ \downarrow \tilde{\iota}_* & & \downarrow \iota_* \\ H_*(\overline{\mathcal{M}}_{g,(2^k)}^{1/5}) & \xrightarrow{\hat{\pi}^*} & H_*(\overline{\mathcal{M}}_{g,\gamma}^{1/5}). \end{array}$$

Due to the dimension reason, $\tilde{\iota}_*[\overline{\mathcal{M}}_{g,(2^k)}^{1/5,5p}]_{\text{loc}}^{\text{vir}} = \sum_j a_j[\xi_j]$, for $a_j \in \mathbb{Q}$ and $\xi_j \in \overline{\mathcal{M}}_{g,(2^k)}^{1/5}$ general such that $\sum a_j = \Theta_{g,k}$. Consequently,

$$\iota_*[\overline{\mathcal{M}}_{g,\gamma}^{1/5,5p}]_{\text{loc}}^{\text{vir}} = \hat{\pi}^* \tilde{\iota}_*[\overline{\mathcal{M}}_{g,(2^k)}^{1/5,5p}]_{\text{loc}}^{\text{vir}} = \sum a_j[\hat{\pi}^{-1}(\xi_j)].$$

Hence using the definition (5.6), we obtain

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_\ell}(\gamma_\ell) \rangle_g^{\text{d.t., } G_5} = \sum a_j \int_{[\hat{\pi}^{-1}(\xi_j)]} \mathbf{e}(\mathcal{L}^\vee)^{-1} \cdot \psi_1^{a_1} \cdots \psi_\ell^{a_\ell}.$$

By Lemma 5.5, we know that

$$\mathbf{c} := \int_{[\hat{\pi}^{-1}(\xi)]} \mathbf{e}_T(\mathcal{L}^\vee) \cdot \psi_1^{a_1} \cdots \psi_\ell^{a_\ell} \in \mathbb{Q}[\mathbf{t}, \mathbf{t}^{-1}]$$

is independent of the choice of the general $\xi \in \overline{\mathcal{M}}_{g,(2^k)}^{1/5}$, and is effectively calculable. Finally applying $\Theta_{g,k} = \sum a_j$, we obtain (5.10).

It remains to look at the case when $(g, k) = (1, 0)$. By stability requirement, we have $\ell \geq 1$. Let $\hat{\pi}$ be the forgetful morphism forgetting all but the first marking. Using Lemma 5.5 and that the Hodge class κ is representable by ψ classes, the correlator (5.10) is a sum of multiples of $\int_{[\overline{\mathcal{M}}_{1,\zeta_5}^{1/5,5p}]_{\text{loc}}^{\text{vir}}} \psi$, and the boundary classes of $\overline{\mathcal{M}}_{1,\zeta_5}^{1/5}$, with the multiplicities explicitly calculable.

The boundary classes can be easily calculated. To calculate $\int_{[\overline{\mathcal{M}}_{1,\zeta_5}^{1/5,5p}]_{\text{loc}}^{\text{vir}}} \psi$, we apply Lemma 5.6. Because $M_0 \rightarrow \overline{\mathcal{M}}_{1,1}$ is quasi-finite and generically a μ_5 -gerbe, $\int_{[M_0]} \psi = \frac{1}{5 \cdot 5 \cdot 24}$, where the additional 5 comes from that ψ is of μ_5 -markings.

To calculate the contribution from $[M_1]$, we need to determine the degree of $M_1 \rightarrow \overline{\mathcal{M}}_{1,1}$. Since $\overline{\mathcal{M}}_{1,\zeta_5}^{1/5} \rightarrow \overline{\mathcal{M}}_{1,1}$ is flat, a μ_5 -gerbe generically, and has 5^2 pre-images over a general point of $\overline{\mathcal{M}}_{1,1}$, the degree we intend to calculate is $5^2 \cdot \frac{1}{5} - \frac{1}{5}$. Thus $\int_{[M_1]} \psi = \frac{25-1}{24 \cdot 5 \cdot 5} = \frac{1}{25}$. This proves the proposition. \square

Lemma 5.8. *For the case $g = 0$, and $\gamma = (1^{k+1}23)$ or $(1^{k+2}4)$ in Proposition 5.3, $\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_\ell}(\gamma_\ell) \rangle_g^{\text{d.t., } G_5}$ is calculable.*

Proof. First, for $\gamma_0 = (123)$ or (1^24) , the moduli $\overline{M}_{0,\gamma_0}^{1/5} \cong B\mu_5$. For $k > 0$, we use the marking forgetful morphism $\overline{M}_{0,\gamma}^{1/5} \rightarrow \overline{M}_{0,\gamma_0}^{1/5} \cong B\mu_5$. Combined with the argument in the previous proof we obtain the lemma. \square

In [CLLL2], we will derive an algorithm that reduces all genus dual-twisted FJRW of the quintic singularity to that of the primitive FJRW invariants. We expect that an analogue of the \mathcal{L} -twisted FJRW invariants of $([\mathbb{C}/\mathbb{Z}_5], x^5)$ (cf. [CZ]) exists for the dual-twisted FJRW invariants of the quintic singularity.

We also need to treat the following term in the localization formula (4.22).

$$(5.11) \quad \int_{[\mathcal{W}_v]_{\text{loc}}^{\text{vir}}} \frac{1}{e_G(R\pi_{v*} ev_v^* \mathcal{O}_{\mathbb{P}^4}(1) \otimes \mathbf{L}_1)} \prod_{i=1}^{n_v} \psi_i^{a_i} ev_i^*(h^{k_i}), \quad k_i \in \mathbb{Z}_{\geq 0},$$

where $h \in H^2(\mathbb{P}^4, \mathbb{Z})$ is the positive generator; v is a stable vertex in $V_0(\Gamma)$; $(\pi_v, ev_v) : \mathcal{C}_v \rightarrow \mathcal{W}_v \times \mathbb{P}^4$ is the universal family of \mathcal{W}_v , and $ev_i : \mathcal{W}_v \rightarrow \mathbb{P}^4$ is the evaluation at i -th markings.

Lemma 5.9. *The integral (5.11) is a monomial in \mathfrak{t} , of the form*

$$(-1)^{d_v+1-g_v} \cdot \langle \tau_{a_1}(h^{k_1}) \cdots \tau_{a_{n_v}}(h^{k_{n_v}}) \rangle_{g_v, d_v}^{GW} \cdot \mathfrak{t}^{d_v+1-g_v}.$$

Here $\langle \cdots \rangle_{g,d}^{GW}$ is the genus- g degree- d GW invariant of Q_5 , defined as

$$(5.12) \quad \langle \tau_{a_1}(h^{k_1}) \cdots \tau_{a_n}(h^{k_n}) \rangle_{g,d}^{GW} = \int_{[\overline{M}_{g,n}(Q_5,d)]^{\text{vir}}} \prod_{i=1}^n \psi_i^{a_i} ev_i^*(h^{k_i}).$$

Proof. After fixing an ordering of E_v , we have isomorphism $\mathcal{W}_v \cong \overline{\mathcal{M}}_{g_v, n_v}(\mathbb{P}^4, d)^p$, and $[\mathcal{W}_v]_{\text{loc}}^{\text{vir}} = [\overline{\mathcal{M}}_{g_v, n_v}(\mathbb{P}^4, d)^p]_{\text{loc}}^{\text{vir}}$. By [CL1],

$$(5.13) \quad [\overline{\mathcal{M}}_{g_v}(\mathbb{P}^4, d)^p]_{\text{loc}}^{\text{vir}} = (-1)^{d_v+1-g_v} [\overline{\mathcal{M}}_{g_v}(Q_5, d_v)]^{\text{vir}} \in A_0 \overline{\mathcal{M}}_{g_v}(Q_5, d_v).$$

Let $\theta : \overline{\mathcal{M}}_{g_v, n_v}(Q_5, d_v) \rightarrow \overline{\mathcal{M}}_{g_v}(Q_5, d_v)$ be the forgetful map. Then we have

$$\theta^* [\overline{\mathcal{M}}_{g_v}(\mathbb{P}^4, d_v)^p]_{\text{loc}}^{\text{vir}} = [\overline{\mathcal{M}}_{g_v, n_v}(\mathbb{P}^4, d_v)^p]_{\text{loc}}^{\text{vir}}.$$

The lemma then follows from a direct calculation. \square

Remark 5.10. Using string and dilaton equations, (5.12) can be calculated knowing the GW invariants of quintics $N_{g,d}$.

5.3. FJRW invariants lead to GW-invariants. In this subsection we use MSP moduli spaces to produce an algorithm calculating the GW invariants of the quintic from the FJRW invariants.

Theorem 5.11. Let $g \geq 1$, $d \geq 1$ be such that $d + 1 - g > 0$. Then the vanishing (1.2) produces a relation evaluating $N_{g,d}$, provided the following are known:

- (1) $N_{g',d'}$ for $g' \leq g$ and $d' \leq d - 1$, and
- (2) $\Theta_{g,k}$ for $k \leq 2g - 2$ and $\Theta_{g',k}$ for $g' \leq g - 1, k \leq 2g - 4$.

Proof. We first prove the case $g \geq 2$. Choose $\mathbf{d} = (d, 0)$ and $\ell = 0$, and consider the moduli of stable MSP fields $\mathcal{W}_{g,\mathbf{d}}$ of numerical data $(g, \emptyset, \mathbf{d})$. We apply the vanishing (1.2) to the equivariant cycle $[\mathcal{W}_{g,\mathbf{d}}]_{\text{loc}}^{\text{vir}}$ to obtain

$$(5.14) \quad 0 = \sum_{\Gamma \in \Delta_{g,\mathbf{d}}^{\text{reg}}} \text{Contr}(\Gamma), \quad \text{Contr}(\Gamma) = \left[\mathbf{t}^{\delta(g,\mathbf{d})} \cdot \frac{[(\mathcal{W}_{g,\mathbf{d}})_{(\Gamma)}^T]_{\text{loc}}^{\text{vir}}}{e(N_{(\mathcal{W}_{g,\mathbf{d}})_{(\Gamma)}^T/\mathcal{W}_{g,\mathbf{d}})}} \right]_0.$$

For $\Gamma \in \Delta_{g,\mathbf{d}}^{\text{reg}}$, applying Proposition 3.5 we see that $\text{Contr}(\Gamma)$ is a polynomial expression in terms of the twisted GW of Q_5 and the dual-twisted FJRW invariants of $([\mathbb{C}^5/\mathbb{Z}_5], \mathfrak{w}_5)$. By the previous subsections, they can all be expressed in terms of $N_{g',d'}$ and $\Theta_{g',k'}$.

We now investigate possible $N_{g',d'}$ and $\Theta_{g',k'}$ that appear in $\text{Contr}(\Gamma)$. By Proposition 3.5 and the previous discussion, we see that each $v \in V_0^S(\Gamma)$ contributes an entry N_{g_v, d_v} in $\text{Contr}(\Gamma)$, and each $v \in V_\infty^S(\Gamma)$ contributes an entry Θ_{g_v, k_v} in $\text{Contr}(\Gamma)$, where k_v is the number of $e \in E_v$ so that the monodromy of $\mathcal{L}|_{\mathcal{C}_v}$ along $y_{(e,v)} = \mathcal{C}_e \cap \mathcal{C}_v$ is ζ_5^2 .

We first show that if $N_{g',d'}$ appears in $\text{Contr}(\Gamma)$, then $g' \leq g$ and $d' \leq d$, and if $N_{g,d}$ appears in $\text{Contr}(\Gamma)$, then Γ is a one vertex graph. Indeed, because the MSP fields all have domain curves genus g , and $g = \sum_{v \in V^S(\Gamma)} g_v + h^1(\Gamma)$, we have $g_v \leq g$ for all $v \in V^S(\Gamma)$. Furthermore when $g_v = g$ for one $v \in V_0$ then all other $g_{v'} = 0$ and $h^1(\Gamma) = 0$. For the degree bound, we recall that (cf. [CLLL]) by the non-vanishing properties of ν_1 and ν_2 , both $\mathcal{N} \otimes \mathcal{L}|_{\mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty}$ and $\mathcal{N}|_{\mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1}$ are trivial line bundles. Thus using $\deg \mathcal{N} = d_\infty = 0$, we

get $\deg \mathcal{L}|_{\mathcal{C}_0 \cup \mathcal{C}_{01}} = d_0 = d$. Because $\deg \mathcal{L}|_{\mathcal{C}_e} > 0$ for every $e \in E_0(\Gamma)$, we have $d_v = \deg \mathcal{L}|_{\mathcal{C}_v} \leq d$. Finally, suppose $N_{g',d}$ appears in $\text{Contr}(\Gamma)$, then $E_0(\Gamma) = \emptyset$ and $V_0^S(\Gamma) \neq \emptyset$. Since Γ is connected and regular, Γ must be a single vertex graph, say $\{v\} = V_0^S(\Gamma)$, and then $g_v = g$. For such graph, we have $\text{Contr}(\Gamma) = N_{g,d}$. This proves (1).

We now look at the possible $\Theta_{g',k'}$ appearing in $\text{Contr}(\Gamma)$. For $v \in V_\infty(\Gamma)$, we adopt the convention that

$$\mathcal{C}_{[v]} = \mathcal{C}_v \cup \left(\bigcup_{e \in E_v} \mathcal{C}_e \right),$$

which is a union of \mathcal{C}_v with all \mathcal{C}_e 's intersecting \mathcal{C}_v . Let $V_{\text{exc}}(\Gamma)$ be the set of $v \in V_\infty^S(\Gamma)$ that are exceptional (c.f. Section 2.5), namely $v \in V_\infty^S(\Gamma)$ with $g_v = 0$ and $\gamma_v = (1^{2+k}4)$ or $(1^{1+k}23)$. For each $v \in V_{\text{exc}}^S(\Gamma)$,

$$(5.15) \quad \deg \mathcal{N}|_{\mathcal{C}_{[v]}} = -(d_v + \sum_{e \in E_v} d_e) \geq -\left(\frac{k+1}{5} - \frac{6+k}{5}\right) = 1.$$

For each v in $V_\infty^S(\Gamma) - V_{\text{exc}}(\Gamma)$, since Γ is regular, for every $e \in E_v$, $\gamma_{(e,v)} = \zeta_5$ or ζ_5^2 . If $\gamma_{(e,v)} = \zeta_5$, $d_e \leq -1/5$, and if $\gamma_{(e,v)} = \zeta_5^2$, $d_e \leq -2/5$. Let s_v (resp. k_v) be the number of $e \in E_v$ whose $\gamma_{(e,v)} = \zeta_5$ (resp. ζ_5^2). Adding that both $\nu_1|_{\mathcal{C}_v}$ and $\rho|_{\mathcal{C}_v}$ are nowhere vanishing, we have

$$\deg \mathcal{N}^{\vee \otimes 5}|_{\mathcal{C}_v} = \deg \omega_{\mathcal{C}_v} \left(\sum_{e \in E_v} y_{(e,v)} \right) = (2g_v - 2 + |E_v|).$$

Therefore, $5 \deg \mathcal{N}|_{\mathcal{C}_v} = -(2g_v - 2) - |E_v|$.

Using (5.15), that $\mathcal{N}|_{\mathcal{C}_1 \cup \mathcal{C}_{01} \cup \mathcal{C}_0}$ is trivial and $\deg \mathcal{N}|_{\mathcal{C}_{[v]}} > 0$ for unstable $v \in V_\infty$, we have

$$\begin{aligned} 0 = d_\infty &= \sum_{v \in V_\infty(\Gamma)} \deg \mathcal{N}|_{\mathcal{C}_{[v]}} \geq \sum_{v \in V_\infty^S(\Gamma) - V_{\text{exc}}(\Gamma)} \deg \mathcal{N}|_{\mathcal{C}_{[v]}} \\ &\geq \sum_{v \in V_\infty^S(\Gamma) - V_{\text{exc}}(\Gamma)} \frac{k_v - (2g_v - 2)}{5}. \end{aligned}$$

Let $v \in V_\infty^S(\Gamma)$. First we have $g_v \leq g$. If $g_v = g$, then all other $g_{v'} = 0$. Thus the above inequality shows that $k_v \leq 2g_v - 2$. This proves the first part of (2).

We prove the second part of (2). If $g_v \leq g - 1$ for all $v \in V_\infty^S(\Gamma)$, then $\sum_{v \in V_\infty^S(\Gamma) - V_{\text{exc}}(\Gamma)} (2g_v - 2) \leq 2g - 4$. Combined with the previous inequality, we obtain $k_v \leq 2g - 4$ for every $v \in v \in V_\infty^S(\Gamma) - V_{\text{exc}}(\Gamma)$. This proves (2).

Finally, the contribution from $v \in V_1^S(\Gamma)$ are integrals of ψ classes on $\overline{\mathcal{M}}_{g',n'}$, which can be effectively calculated.

Combined, this shows that (5.14) is a polynomial expression of terms as stated in (1) and (2) with coefficients involving Hodge integrals on $\overline{\mathcal{M}}_{g',n'}$ and other calculable terms. This proves the case $g \geq 2$.

In case $g = 1$, a similar argument, using Proposition 5.7, shows that we can determine all $N_{1,d}$, knowing the terms specified in the theorem. This proves the theorem. \square

Remark 5.12. For $g = 2$, to run this algorithm we need to know $N_{2,1}$ and $\Theta_{2,2}$. As $N_{2,1}$ is classical, only $\Theta_{2,2}$ is unknown. Similarly, for $g = 3$ we only need $\Theta_{2,2}$ and $\Theta_{3,4}$ since $N_{3,1}$ and $N_{3,2}$ are classical. (Recall that $\Theta_{g,k} = 0$ unless $k + 2 - 2g \equiv 0(5)$.)

5.4. Algorithm for FJRW invariants. We pick d so that $d + 1 - g > 0$ and let $\mathbf{d} = (0, d)$, then it is easy to see that no GW invariants $N_{d',g'}$ appears in the polynomial relation from the vanishing (1.2) for $(g, \gamma = \emptyset, \mathbf{d})$. This provides relations among FJRW invariants $\Theta_{g',k'}$.

Theorem 5.13. For fixed $g \geq 1$, and $k \geq 7g - 2$, the vanishings (1.2) using $(g, \gamma, \mathbf{d}) = (g, \emptyset, (0, d))$ (for some $d > g - 1$) evaluate $\Theta_{g,k}$ inductively from the datum $\{\Theta_{g',k'}\}_{g' < g, k' \leq k}$ and $\{\Theta_{g,k'}\}_{k' < 7g-2}$.

Proof. We calculate $\Theta_{g,k}$ for $k \geq 7g - 2$. As $\Theta_{g,k} = 0$ unless $5 \mid k - 7g + 2$, we only need to examine when $k = 7g - 2 + 5m$, $m \geq 0$. Pick $d = g + m$ and let $\mathbf{d} = (0, d)$. Applying the vanishing (1.2) to cycle $[\mathcal{W}_{g,\mathbf{d}}]_{\text{loc}}^{\text{vir}}$, as (5.14), we obtain

$$\sum_{\Gamma \in \Delta_{g,\mathbf{d}}^{\text{reg}}} \text{Contr}(\Gamma) = 0.$$

For each $\Gamma \in \Delta_{g,\mathbf{d}}^{\text{reg}}$, $\text{Contr}(\Gamma)$ is a polynomial expression in terms of a collection of $N_{g',d'}$ and $\Theta_{g',k'}$. Since $\mathcal{N} \otimes \mathcal{L}|_{\mathcal{C}_1 \cup \mathcal{C}_{1\infty} \cup \mathcal{C}_\infty}$ and $\mathcal{N}|_{\mathcal{C}_0 \cup \mathcal{C}_{01} \cup \mathcal{C}_1}$ are trivial, we have $\deg \mathcal{L}|_{\mathcal{C}_0 \cup \mathcal{C}_{01}} = d_0$. As the degree of \mathcal{L} is nonnegative on each component of \mathcal{C}_0 and positive on each component of \mathcal{C}_{01} , the assumption $d_0 = 0$ implies $V_0(\Gamma) = E_0(\Gamma) = \emptyset$. Therefore no $N_{g',d'}$ occurs.

As argued before, only $\Theta_{g' \leq g, k'}$ can possibly appear in $\text{Contr}(\Gamma)$. We look at possible $\Theta_{g,k'}$ appearing in it. Following the notation in the proof of Theorem 5.11, we get

$$\begin{aligned} (5.16) \quad g + m = d_\infty &= \sum_{v \in V_\infty(\Gamma)} \deg \mathcal{N}|_{\mathcal{C}_{[v]}} = \sum_{v \in V_\infty^S(\Gamma) - V_{\text{exc}}(\Gamma)} \frac{k_v - (2g_v - 2)}{5} \\ &+ \sum_{v \in V_{\text{exc}}(\Gamma)} \deg \mathcal{N}|_{\mathcal{C}_{[v]}} + \sum_{v \in V_\infty^U(\Gamma)} \deg \mathcal{N}|_{\mathcal{C}_{[v]}} \\ &\geq \sum_{v \in V_\infty^S(\Gamma) - V_{\text{exc}}(\Gamma)} \frac{k_v - (2g_v - 2)}{5}. \end{aligned}$$

If $g_v < g$ for every $v \in V_\infty^S(\Gamma)$, then we have a bound

$$\sum_v (k_v + 2) \leq 5g + 5m + 2 \sum_v g_v \leq 7g + 5m = k + 2.$$

This gives the bound $k_v \leq k$.

Next we consider the case when there is a $v \in V_\infty^S(\Gamma)$ so that $g_v = g$. Then all other $v' \in V_\infty^S(\Gamma)$ has $g_{v'} = 0$. Therefore the above inequality implies $k_v \leq 7g - 2 + 5m = k$. Now assume the equality holds, i.e., $k_v = 7g - 2 + 5m$. From (5.15), we have $\deg \mathcal{N}|_{\mathcal{C}_{[v]}} > 0$ for $v \in V_{exc}(\Gamma)$. We obviously also have $\deg \mathcal{N}|_{\mathcal{C}_{[v]}} > 0$ for $v \in V_\infty^U(\Gamma)$. Since all inequalities in the formula (5.16) become equality, we have $V^S(\Gamma) = V_\infty^S(\Gamma) = \{v\}$, and $E_v = \{e_1, \dots, e_k\}$ with each $d_{e_i} = -2/5$. By Proposition 3.5

$$[\mathcal{W}(\Gamma)]_{\text{loc}}^{\text{vir}} = \frac{1}{k!} \frac{1}{2^k} (\iota_\Gamma^-)_* \left([\overline{\mathcal{M}}_{g, (2k)}^{1/5, 5p}]_{\text{loc}}^{\text{vir}} \right).$$

We calculate $\text{Contr}(\Gamma)$. Recall $\mathfrak{e}_T(\mathcal{L}^\vee) = e_{\mathbb{G}_m}(R\pi_* \mathcal{L}^\vee \otimes \mathbf{L}_{-1})$ (cf. (5.5)). Applying Theorem 4.9 and the formulas in the previous section, we get

$$\begin{aligned} \text{Contr}(\Gamma) &= \left[\mathfrak{t}^{\delta(g, \mathbf{d})} \cdot \frac{[(\mathcal{W}_{g, \mathbf{d}})^T]_{\text{loc}}^{\text{vir}}}{e(N_{(\mathcal{W}_{g, \mathbf{d}})^T(\Gamma)/\mathcal{W}_{g, \mathbf{d}}})} \right]_0 \\ &= \left[\mathfrak{t}^{1+m} (5\mathfrak{t} \cdot \frac{5\mathfrak{t}}{2})^k \cdot (\frac{1}{\frac{5\mathfrak{t}}{2} \cdot 5\mathfrak{t}})^k \cdot \frac{[\overline{\mathcal{M}}_{g, \gamma}(G_5)^p]_{\text{loc}}^{\text{vir}}}{k! 2^k \mathfrak{e}_T(\mathcal{L}^\vee)} \frac{1}{\prod_{i=1}^k (-\frac{\mathfrak{t}}{2} - \psi_i)} \right]_0 \\ &= \mathfrak{t}^{1+m} \frac{\Theta_{g, k}}{k! 2^k (-\mathfrak{t})^{3-4m-7g}} \cdot \frac{1}{(-\frac{\mathfrak{t}}{2})^k} = (-1)^{m+1} (k!)^{-1} \Theta_{g, k}, \end{aligned}$$

where we have used $\text{vir. dim} [\overline{\mathcal{M}}_{g, \gamma}(G_5)^p]_{\text{loc}}^{\text{vir}} = 0$ and $\text{rank } R\pi_* \mathcal{L}^\vee = 3 - 4m - 7g$.

This proves that the vanishing $0 = \sum_{\Gamma \in \Delta_{g, \mathbf{d}}^{\text{reg}}} \text{Contr}(\Gamma)$ expresses $\Theta_{g, k}$ as a polynomial of invariants indicated in the statement of the theorem. \square

Corollary 5.14. *The collection $\{\Theta_{1, k}\}_k$ can be determined effectively. The collection $\{\Theta_{2, k}\}_k$ can be determined effectively once $\Theta_{g=2, 2}$ and $\Theta_{g=2, 7}$ are known. The set $\{\Theta_{3, k}\}_k$ can be determined effectively once the invariants $\Theta_{g=3, 4}$, $\Theta_{g=3, 9}$, $\Theta_{g=3, 14}$, $\Theta_{g=2, 2}$ and $\Theta_{g=2, 7}$ are known.*

5.5. A simple example. Here we illustrate how to use various MSP moduli spaces to calculate invariants via an example. We use $\gamma = (\rho)$ to mean one marking labeled by $(1, \rho)$. We first use the vanishing

$$(\mathfrak{t} \cdot [\mathcal{W}_{1, (\rho), (0, 0)}]_{\text{loc}}^{\text{vir}})_0 = 0.$$

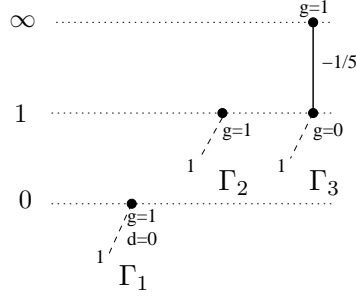
Via localization, we obtain

$$\text{Contr}(\Gamma_1) + \text{Contr}(\Gamma_2) + \text{Contr}(\Gamma_3) = 0,$$

where $\Gamma_1, \Gamma_2, \Gamma_3$ are shown in Figure 1 below.

Using the localization formula in Section 4, we compute that $\text{Contr}(\Gamma_1) = \frac{25}{3}$, and $\text{Contr}(\Gamma_2) = \frac{1}{5}$. Therefore,

$$\text{Contr}(\Gamma_3) = -\frac{25}{3} - \frac{1}{5} = -\frac{128}{15}.$$

FIGURE 1. Regular labelled graphs for $g = 1$, $\gamma = (\rho)$, $\mathbf{d} = (0, 0)$.

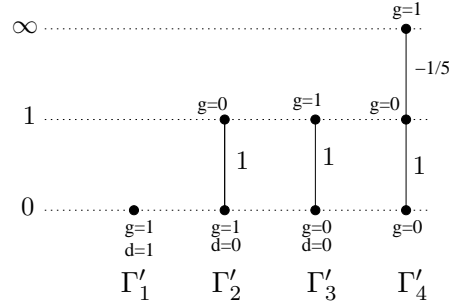
We next use the vanishing

$$[\mathfrak{t} \cdot \mathcal{W}_{1, \emptyset, (1, 0)}]_{\text{loc}}^{\text{vir}} = 0.$$

Via localization, we obtain

$$\text{Contr}(\Gamma'_1) + \cdots + \text{Contr}(\Gamma'_4) = 0,$$

where $\Gamma'_1, \dots, \Gamma'_4$ are shown in Figure 2 below.

FIGURE 2. Regular labelled graphs for $g = 1$, $\gamma = \emptyset$, $\mathbf{d} = (1, 0)$.

We compute that

$$\text{Contr}(\Gamma'_1) = -N_{1,1}, \quad \text{Contr}(\Gamma'_2) = \frac{9625}{6}, \quad \text{Contr}(\Gamma'_3) = -\frac{4087}{12}.$$

Comparing $\text{Contr}(\Gamma'_4)$ with $\text{Contr}(\Gamma_3)$, we obtain

$$\text{Contr}(\Gamma'_4) = 120 \cdot \text{Contr}(\Gamma_3) = -1024.$$

Therefore,

$$N_{1,1} = \frac{9625}{6} - \frac{4087}{12} - 1024 = \frac{2875}{12}.$$

5.6. Conjectural algorithm. Our goal is to determine the set of all quintic GW invariants $\{N_{g,d}\}_{g,d}$ and the FJRW invariants $\{\Theta_{g,k}\}_{g,k}$. As all genus zero invariants $\{N_{0,d}, \Theta_{0,d}\}_d$ are known by [Gi, LLY, CR], we only need to find algorithms to determine all positive genus invariants. Inductively, suppose we have algorithms determining all $N_{g'<g,d}$ and $\Theta_{g'<g,k}$, Theorem 5.11 shows that to determine all genus g invariants we only need to find additional algorithms to determine the sets $\{N_{g,d}\}_{d\leq g-1}$ and $\{\Theta_{g,k}\}_{k<2g-2}$. Together, we need to determine the following two types of initial datum

$$(5.17) \quad I_1 = \{N_{g,d} \mid d < g\} \quad \text{and} \quad I_2 = \{\Theta_{g,k} \mid k < 7g - 2\}$$

For the set I_1 , say for $d < g$, let $\mathbf{d} = (d+1, g-d-1)$ and $\gamma = \emptyset$.

Conjecture 5.15. *For $d < g$, using the mentioned (g, γ, \mathbf{d}) and applying the vanishing in (1.2), we can determine $N_{g,d}$ provided*

- (1) $N_{g',d'}$ are known for $g' < g$, $d' \leq d$, and
- (2) $\Theta_{g',k'}$ are known for $g' \leq g$, $k' < 7g' - 2$.

Note that Conjecture 5.15 implies Conjecture 1.3 via Theorem 5.11. Suppose Conjecture 5.15 is confirmed, we are left to determine I_2 . To this end, we propose to use non-trivial γ . Specifically, we let $\ell > 0$ and consider $\gamma = ((1, \rho)^\ell)$, with ℓ markings labeled by $(1, \rho)$. (We abbreviate it to $\gamma = (\rho^\ell)$.) For positive k so that $d = (k+2-2g)/5 \in \mathbb{Z}$ but $d+1-g \leq 0$, we pick $\ell \geq 0$ so that

$$\text{vir. dim } \mathcal{W}_{g, \rho^\ell, (0, d_k)} = \ell + d + 1 - g = 1 > 0.$$

We then apply the vanishing (1.2) to the moduli $\mathcal{W}_{g, (\rho^\ell), (0, d)}$. As $d_0 = 0$, this vanishing provides a polynomial relation among $\{\Theta_{g', \leq g, k'}\}$.

Like before, among all $\Theta_{g, k'}$ appearing in this polynomial relation, the largest possible k' is $\Theta_{g, k}$, and it appears in the form $e_{g, k} \Theta_{g, k}$, where $e_{g, k}$ is a scalar. If $e_{g, k} \neq 0$, then we can evaluate $\Theta_{g, k}$ using this relation. This proves

Proposition 5.16. *Let the notation as stated. Then in case $e_{g, k} \neq 0$, the stated algorithm evaluates $\Theta_{g, k}$ based on $\{\Theta_{g', k'}\}_{g' < g, k' \leq k}$, and $\{\Theta_{g, k'}\}_{k' < k}$*

Conjecture 5.17. *The constant $e_{g, k} \neq 0$.*

Though the coefficient $e_{g, k}$ theoretically is calculable, it is quite involved. For the case $g = 1$ and $k = 0$, we verified that $e_{1, 0} \neq 0$, and using this we calculated $\deg[\overline{M}_{1, \zeta_5}^{1/5, 5p}]_{\text{loc}}^{\text{vir}}$. This number together with the algorithm in Theorem 5.11 calculates the quintic's GW $N_{1, 1} = \frac{2875}{12}$ (cf. the previous subsection). We also confirmed that $e_{2, 2} \neq 0$.

We end this subsection with a simple observation. In [BCOV, HKQ], assuming the mirror symmetry for all genus and using “holomorphic anomaly equation”, the genus g GW generating function $F_g = \sum N_{g, d} q^d$ can be determined based on the first $3g - 2$ many $N_{g, d}$'s. Later, using the *gap condition* at

the conifold expansion and the *monodromy* at the orbifold expansion, [HKQ, Sect 3.6] further reduced the number of unknowns to $\lfloor \frac{2g-2}{5} \rfloor$.

In our approach, granting Conjecture 5.15, the number of initial data needed to determine F_g are the FJRW invariants $\Theta_{g,k \leq 2g-2}$, subject to $2g-2 \equiv k(5)$. Thus $\lfloor \frac{2(g-1)}{5} \rfloor + 1$ many FJRW invariants are needed to determine $N_{g,d}$ via MSP moduli, provided all lower genus invariants are known.

5.7. Witten's GLSM theory. In [Wi], Witten introduced a family of theories using path integrals, the Gauged Linear Sigma Model (GLSM), linking a sigma model targeting a Calabi-Yau hypersurface to a sigma model targeting a Landau-Ginzburg space. The GLSM is parameterized by a complex (phase) parameter $t = ir + \frac{\theta}{2\pi}$, where r is the coefficient of the ‘‘Fayet-Iliopoulos D term’’, and θ is called theta-angle (c.f. [Wi, page 12 and (2.28)]). Witten postulated that the GLSM specializes to GW path integral when r approaches infinity, and specializes to LG model path integral when r approaches negative infinity. This is the CY/LG correspondence commonly referred to.

Our Mixed-Spin-P fields (MSP fields) is a field version of ‘‘phase space transition’’. An MSP field is an interpolation between fields valued in $K_{\mathbb{P}^4}$ and fields valued in $[\mathbb{C}^5/\mathbb{Z}_5]$. The interpolation is governed by the ‘‘ ν field’’. Over the part of worldsheet (curve) where $\nu = 0$, the MSP field is a pure field taking values in $K_{\mathbb{P}^4}$, and over $\nu = \infty$ the MSP field is a pure field taking values in $[\mathbb{C}^5/\mathbb{Z}_5]$. By replacing the phase parameter by a field, we transform Witten's family of CY/LG correspondence into a single field theory. In this sense, the field ν is our attempt to ‘‘quantize’’ Witten's parameter $t = ir + \frac{\theta}{2\pi}$ in his vision of Calabi-Yau/Landau/Ginzberg correspondence.

5.8. Comparing with other approaches. The other approach to Witten's proposal is to develop a general theory of GLSM (gauged linear σ -model). The work of Fan, Jarvis and Ruan [FJR3] worked out this theory in narrow case, using cosection localized virtual cycle technique. The work of Tian and Xu [TX1, TX2] are to develop an analytic theory of broad GLSM theory.

In [FJR3, Example 4.2.23], a special case of GLSM theory was exhibited, where a closed point of this moduli space consists of a pointed twisted curve $\Sigma \subset \mathcal{C}$, two line bundles \mathcal{L} and \mathcal{N} , and a collection of sections. Though this is kind of similar to the MSP fields introduced in [CLLL] and studied in the current paper, the [FJR3, Example 4.2.23] and the theory of MSP fields are different.

Specifically, for the $\epsilon = 0^+$ moduli space in [FJR3, Example 4.2.23], (the case for GW-theory is when $\epsilon = +\infty$, yet to be constructed,) the stability (of a point $(\mathcal{C}, \mathcal{L}, \mathcal{N}, \dots)$) requires that $\mathcal{L}^{-e_1} \otimes \mathcal{N}^{-e_2}$ ($0 < e_1 < e_2$) is ample on components of \mathcal{C} of which $\omega_{\mathcal{C}}^{\log}$ has degree zero. We show that there are stable MSP fields which are not $\epsilon = 0^+$ stable. For example, the graph in Figure 3 has an edge connecting a genus 1 curve with a genus zero curve; an MSP field whose graph is as in Figure 3 will have an irreducible component E satisfying

$\deg(\omega_c^{\log}|_E) = 0$, $\mathcal{N}|_E \cong \mathcal{O}$ and $\deg(\mathcal{L}|_E) = 1$, thus $\deg((\mathcal{L}^{-e_1} \otimes \mathcal{N}^{-e_2})|_E) < 0$. Such MSP fields will not be in the $\epsilon = 0^+$ moduli space of [FJR3, Ex.4.2.23].

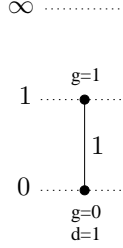


FIGURE 3.

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MATHEMATICS DEPARTMENT, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

MATHEMATICS DEPARTMENT, STANFORD UNIVERSITY, AND SHANGHAI CENTER FOR MATHEMATICAL SCIENCE, FUDAN UNIVERSITY

MATHEMATICS DEPARTMENT, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

MATHEMATICS DEPARTMENT, COLUMBIA UNIVERSITY